Engineering Mathematics
2023; 7(1): 1-18
http://www.sciencepublishinggroup.com/j/engmath
doi: 10.11648/j.engmath.20230701.11
ISSN: 2640-0855 (Print); ISSN: 2640-088X (Online)

Relative Controllability for a Class of Linear Impulsive Systems

Gwang Jin Kim[1], Su Song Om[1], Tae Gun Oh[1], Nam Chol Yu[2, *], Jin Sim Kim[3]

[1]Faculty of Mathematics, Kim Hyong Jik University of Education, Pyongyang, Democratic People's Republic of Korea
[2]School of Science and Engineering, Kim Chaek University of Technology, Pyongyang, Democratic People's Republic of Korea
[3]International Technology Cooperation Center, Kim Chaek University of Technology, Pyongyang, Democratic People's Republic of Korea

Email address:
ync781213@star-co.net.kp (Nam Chol Yu)
[*]Corresponding author

To cite this article:
Gwang Jin Kim, Su Song Om, Tae Gun Oh, Nam Chol Yu, Jin Sim Kim. Relative Controllability for a Class of Linear Impulsive Systems. *Engineering Mathematics*. Vol. 7, No. 1, 2023, pp. 1-18. doi: 10.11648/j.engmath.20230701.11

Received: November 3, 2022; **Accepted**: December 15, 2022; **Published**: February 27, 2023

Abstract: Hybrid systems are systems that involve continuous and discrete event dynamical behaviors. A impulsive system is a special hybrid system. The continuous dynamics of impulsive systems are usually described by ordinary differential equations and the discrete event dynamics with instantaneously rapid jumps are described by switching laws. Various complex dynamical phenomena that can be modeled by impulsive systems arise in many areas of modern science and technology such as economics, physics, chemistry, biology, information science, radiotherapy, acupuncture, robotics, neural networks, automatic control, artificial intelligence, space technology, and telecommunications, etc. In modern control theory, controllability is one of the most important dynamical properties of considered impulsive systems, therefore, the controllability problem is regarded as one of the fundamental issues of impulsive systems. The basic questions for controllability of impulsive systems as well as for the ordinary systems without impulses and with control function are to obtain useful criteria that allow us to identify whether given dynamic systems are controllable or not. Up to now there have been being many investigation results for controllability of different kinds of impulsive systems with respect to the terminal state constraints of a point type. The purpose of this paper is to study relative controllability with respect to the terminal state constraint of a general type for a class of linear time-varying impulsive systems. In this paper, several types of criteria for relative controllability of such systems are established by a algebraic method, that is, specially speaking, by the matrix rank method. Some corresponding necessary and sufficient conditions for controllability of linear time-invariant impulsive systems are also obtained more compactly. Meanwhile, for given impulsive systems some equivalent relationships between different kinds of controllability are investigated and our criteria for relative controllability are compared with the existing results. A simple example is given to illustrate the utility of our criteria.

Keywords: Impulsive Systems, Impulsive Control, Complete Controllability, Null Controllability, Relative Controllability, Relative Null Controllability

1. Introduction

Each impulsive system is characterized by a certain combination of continuous evolution and discrete transition and sometimes is called special discrete-continuous system [9]. As mentioned in the literature of modern control theory, most processes have been described and analyzed by impulsive control systems (see, e.g., [11, 13, 16, 19] and references therein). Therefore, it is very important and actual to more actively study impulsive control systems and the study has been receiving increasing interest in the control community recently due to its theoretical challenges and practical significances in many real world applications. In particular, it is worth to note that the study of controllability problems plays an important role in the whole control theory and in this area fundamental issue is to obtain criteria for controllability of various systems. Nowadays, most efforts have been focused on the research of criteria for controllability of impulsive systems by using different approaches. But differently from continuous systems, till now, there are comparatively few results on controllability for various kinds of impulsive systems.

The study for controllability of impulsive systems first had begun by the work [7] of Leela in 1993 and then a significant progress has been being made in the past three decades [5, 10, 11, 16, 18-20]. In 1995 when impulsive control was yet to become popular, Liu [8] investigated the necessary and sufficient conditions of complete controllability for a class of linear impulsive system only with impulse actions without continuous control. In the literature there have been some serious investigative results undertaken dealing with necessary and sufficient conditions for controllability of impulsive systems. However, greatly to our regret, such developments have been being made mainly in terms of null and/or complete controllabilities. Complete controllability of impulsive systems have been extensively studied by many researches, for instance, see works [2, 6, 8, 11, 19] and references therein. On the aspects for null controllability of impulsive systems we refer to works [1, 3, 4, 7, 12-14, 17, 18] and references therein. In particular, in 2010 Zhao and Sun [18] investigated the sufficient and necessary conditions for null controllability of linear impulsive systems in complex fields, inspired by [3] (2002) and [17] (2009), where the authors considered the fundamental concepts of null controllability of real linear time-varying impulsive systems by an algebraic approach. It was noted by George in [2] (2000) that generally speaking, the concepts "null controllability" and "complete controllability" are not equivalent for impulsive systems. There has been an increasing interest in the investigation for relative controllability of impulsive systems with the right end state in one point over the state space (in the origin or other), however, there are still very few results, see for instance [16] (2022) and the references therein. Moreover, to the best of our knowledge, there are not results for relative controllability of impulsive systems with respect to the more general right terminal state constraints.

The main purpose of this paper is to drive necessary and sufficient criteria for relative controllability of linear time-varying and linear time-invariant impulsive systems with respect to linear right terminal state constraint of a type.

The rest of this paper is organized as follows. In Section 2 we give some notations, concepts and hereafter required lemmas. The main results for relative controllability of the linear time-varying impulsive system are given in Section 3. New criteria for relative controllability of linear time-invariant impulsive system are obtained in Section 4. Furthermore, in Sections 3, 4, several equivalent relationships between the different kinds of controllability are also established. Section 5 contains a simple example to illustrate the utility of obtained results. Finally, we provide the conclusion in Section 6.

2. Preliminaries

Consider the linear time-varying impulsive system

$$\begin{aligned} &\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \neq t_i \\ &x(t_i^+) = x(t_i^-) + D(t_i)v_i, \quad t = t_i \\ &x(t_0^+) = x_0, \ \ Hx(t^*) = g, \end{aligned} \tag{1}$$

where $A(\cdot):[t_0,t^*]\to R^{n\times n}$ is known $n\times n$ matrix valued continuous function, $B(\cdot):[t_0,t^*]\to R^{n\times r}$, $D(\cdot):[t_0,t^*]\to R^{n\times q}$ are known matrix valued left and piecewise continuous functions respectively, $x_0\in R^n$ is initial state, H is $m\times n$ constant matrix with $rank\ H=m,\ g\in R^m$ is constant vector, $x(t_i^+)$, $x(t_i^-)$ are right, left limits respectively and $x(t_i)=x(t_i^-)$, which implies that the solution of (1) is left continuous at time instant t_i. Note that the initial condition $x(t_0^+)=x_0$ is used rather than $x(t_0)=x_0$. If the initial time t_0 corresponds to a transition time instant then $x(t_0^+)$ is understood to be the initial condition of the ordinary differential equation. The time evolutionary process of impulsive systems is expressed by continuous and jump discontinuous functions. We introduce the following notations for our convenience which we use in the proof of next theorems:

$K:=\{1,\ 2,\ \cdots,r\}$-the set of indices of columns for the matrix $B(t)$

$P:=\{1,\ 2\ ,\ \cdots,q\}$-the set of indices of columns for the matrix $D(t)$.

Now, we make the following assumptions about rest components of impulsive control system (1):

1) Vector valued function $u(\cdot):[t_0,\ t^*]\to R^r$ is piecewise continuous one, which is unknown.
2) For action of impulses $w:=\{(t_i,\ v_i),\ i=1,\ 2,\ \cdots,\ l\}$ natural number l, time instants $t_i\in[t_0,\ t^*]$, $i=1,\ 2,\ \cdots,\ l$ and vectors $v_i\in R^q$, $i=1,\ 2,\ \cdots,\ l$ all are parameters which are unknown.

Given a vector valued piecewise continuous function $u(\cdot):[t_0,\ t^*]\to R^r$ and an action of impulses $w=\{(t_i,\ v_i),\ i=1,\ 2,\ \cdots,\ l\}$, then the pair $\{u(\cdot),\ w\}$ is said to be impulsive control. Corresponding to the impulsive control system (1), we consider the homogeneous system

$$\dot{x}(t) = A(t)x(t),\ x(t_0^+) = x_0. \tag{2}$$

According to ordinary differential equation theory, suppose that $X(t)$ is the fundamental solution matrix of the system (2). Then

$$X(t,\ s) := X(t)X^{-1}(s),\ t,\ s\in\left[t_0,\ +\infty\right)$$

is the transition matrix associated with matrix $A(t)$. It is clear that,

for any $t,\ s,\ \tau \in [t_0,\ +\infty)$,

$$X(t,\ \tau)X(\tau,\ s) = X(t,\ s),\quad X(t,\ t) = I$$

and

$$X(t,\ s) = X^{-1}(s,\ t),$$

where I is the identity matrix of order n [4].

Lemma 2.1. Provided the impulsive control

$$\{u(\cdot),\ w\} = \{u(t),\ t \in [t_0,\ t^*],\ \{(t_i,\ v_i),\ i = 1,\ 2,\ \cdots,\ l\}\},$$

then for any $t \in (t_i,\ t_{i+1}],\ i = 0,\ 1,\ 2,\ \cdots,\ l$, the general solution of system (1) is given by

$$x(t) = X(t,\ t_0)x_0 + \int_{t_0}^{t} X(t,\ s)B(s)u(s)ds + \sum_{j=1}^{i} X(t,\ t_j)D(t_j)v_j \tag{3}$$

where $t_{l+1} := t^*$.

Proof. It follows from the initial condition of system (1) that

$$x(t) = X(t,\ t_0)x_0 + \int_{t_0}^{t} X(t,\ s)B(s)u(s)ds,\quad t \in [t_0,\ t_1],$$

which leads to

$$x(t_1) = X(t_1,\ t_0)x_0 + \int_{t_0}^{t_1} X(t_1,\ s)B(s)u(s)ds.$$

On one hand, since

$$x(t_i^+) = x(t_i^-) + D(t_i)v_i,$$

then

$$x(t_1^+) = X(t_1,\ t_0)x_0 + \int_{t_0}^{t_1} X(t_1,\ s)B(s)u(s)ds + D(t_1)v_1.$$

Next, for $t \in (t_1,\ t_2]$, we have following:

$$\begin{aligned}
x(t) &= X(t,\ t_1)x(t_1^+) + \int_{t_1}^{t} X(t,\ s)B(s)u(s)ds \\
&= X(t,\ t_1)\left[X(t_1,\ t_0)x_0 + \int_{t_0}^{t_1} X(t_1,\ s)B(s)u(s)ds + D(t_1)v_1\right] + \int_{t_1}^{t} X(t,\ s)B(s)u(s)ds = \\
&= X(t,\ t_0)x_0 + \int_{t_0}^{t_1} X(t,\ s)B(s)u(s)ds + X(t,\ t_1)D(t_1)v_1 + \int_{t_1}^{t} X(t,\ s)B(s)u(s)ds = \\
&= X(t,\ t_0)x_0 + \int_{t_0}^{t} X(t,\ s)B(s)u(s)ds + X(t,\ t_1)D(t_1)v_1
\end{aligned}$$

which leads to

$$\begin{aligned}
x(t_2) &= X(t_2,\ t_0)x_0 + \int_{t_0}^{t_2} X(t_2,\ s)B(s)u(s)ds + X(t_2,\ t_1)D(t_1)v_1 \\
x(t_2^+) &= X(t_2,t_0)x_0 + \int_{t_0}^{t_2} X(t_2,s)B(s)u(s)ds + X(t_2,t_1)D(t_1)v_1 + D(t_2)v_2.
\end{aligned}$$

Hence, for $t \in (t_2,\ t_3]$, we have following:

$$x(t) = X(t,\ t_2)x(t_2^+) + \int_{t_2}^{t} X(t,\ s)B(s)u(s)ds$$
$$= X(t,\ t_0)x_0 + \int_{t_0}^{t_2} X(t,\ s)B(s)u(s)ds + X(t,\ t_1)D(t_1)v_1 + X(t,\ t_2)D(t_2)v_2 +$$
$$+ \int_{t_2}^{t} X(t,\ s)B(s)u(s)ds = X(t,\ t_0)x_0 + \int_{t_0}^{t} X(t,\ s)B(s)u(s)ds$$
$$+ X(t,\ t_1)D(t_1)v_1 + X(t,\ t_2)D(t_2)v_2$$

By repeating the procedure as above, we easily can obtain the general solution (3) for $i = 0,\ 1,\ 2,\ \cdots,\ l$. This completes our proof.

Remark 2.1. From the condition $x(t_i) = x(t_i^-)$ and formula (3), we can know that with the impulsive control $\{u(\cdot),\ w\}$, corresponding solution of system (1) is a vector valued piecewise continuous function with left continuity.

Remark 2.2. It is clear that if $D(t) \equiv 0$, that it follows from formula (3) that for ordinary control system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$x(t_0) = x_0$$

the general solution is expressed as

$$x(t) = X(t,t_0)x_0 + \int_{t_0}^{t} X(t,s)B(s)u(s)ds, \quad t \in [t_0,\ t^*],$$

which implies that Lemma 2.1 gives a generalization of the usual solution formula for the continuous linear system, see [4]. In like manner, we verify from the formula (3) that for linear impulsive system without ordinary control

$$\dot{x}(t) = A(t)x(t),\ t \neq t_i$$
$$x(t_i^+) = x(t_i^-) + D(t_i)v_i,\ t = t_i$$
$$x(t_0^+) = x_0$$

the general solution is equal to

$$x(t) = X(t,\ t_0)x_0 + \sum_{j=1}^{i} X(t,\ t_j)D(t_j)v_j, \quad t \in (t_i,\ t_{i+1}],$$
$$i = 0,\ 1,\ 2,\ \cdots,\ l$$

which so too implies that Lemma 2.1 is regarded as a generalization of the result in the reference [6].

The following definition for the relative controllability of the system (1) is adopted in this paper.

Definition 2.1. Impulsive system (1) is said to be relatively state controllable with respect to terminal constraint $Hx(t^*) = g$ on $[t_0,\ t^*]$ (or simply relatively controllable if no confusion), if for any initial state vector $x_0 \in R^n$ and any output vector $g \in R^m$, there exists at least one impulsive control

$$\{u(\cdot),\ w\} = \{u(t),\ t \in [t_0,\ t^*],\ \{(t_i,\ v_i),\ i = 1,\ 2,\ \cdots,\ l\}\}$$

such that with impulsive control $\{u(\cdot),\ w\}$, corresponding solution (trajectory) of system (1) with initial condition $x(t_0^+) = x_0$ satisfies terminal condition $Hx(t^*) = g$.

Definition 2.2. If the above definition 2.1 always let $g = 0$, then impulsive system (1) is said to be relatively state null controllable with respect to terminal constraint $Hx(t^*) = 0$ on $[t_0,\ t^*]$ (or simply relatively null controllable if no confusion).

Remark 2.3. Note that both in the definitions 2.1 and 2.2, if $H = I$, where I is the identity matrix of order n, then we have the definitions of complete and null controllabilities, respectively.

For (1) and some special cases of (1), various sufficient and necessary conditions for complete or null controllabilities of linear impulsive with additional assumptions were studied in the literature, e.g. [4, 6, 8, 15, 18]. It is necessary to recall beforehand the previous results in the references above so that we may verify that our results in this paper are more general and/or new. In [8], the impulsive system (1) with $B(t) \equiv 0$, or only with action of impulses without ordinary control $u(t),\ t \in [t_0,\ t^*]$, namely

$$\begin{aligned} &\dot{x}(t) = Ax(t),\ t \neq t_i \\ &x(t_i^+) = x(t_i^-) + Dv_i,\ t = t_i \end{aligned} \tag{4}$$

where $A,\ D$ are constant matrices, was investigated and the necessary and sufficient condition for complete controllability of system (4) is obtained as follows.

Lemma 2.2. (See [8, Theorem 1].) Impulsive control system (4) is completely controllable on $[t_0,\ t^*]$ if and only if

$$rank\ (D,\ AD,\ A^2D,\ \cdots,\ A^{n-1}D) = n \tag{5}$$

In [4, 18] the impulsive system

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bu(t), t \neq t_i \\ &x(t_i^+) = x(t_i^-) + Du_i, t = t_i \end{aligned} \tag{6}$$

where $A,\ B,\ D$ are $n \times n,\ n \times r,\ n \times r$ constant matrices respectively and instants $t_i \in [t_0, t^*],\ i = 1,\ 2,\ \cdots,\ l$ are fixed, was considered as a special case and sufficient and necessary conditions for null controllability of system (6) are obtained as below.

Lemma 2.3. (See [4, Theorem 3.4] and [18, Theorem 3].) Fixed instants $t_i \in [t_0, t^*],\ i = 1,\ 2,\ \cdots,\ l$, then the following sufficient and necessary null controllability conclusions hold.

i) If

$$rank(B,\ AB,\ A^2B,\ \cdots,\ A^{n-1}B) = n \tag{7}$$

then impulsive system (6) is null controllable on $[t_0,\ t^*]$.

ii) If impulsive system (6) is null controllable on $[t_0,\ t^*]$, then

$$rank(B,\ AB,\ A^2B,\ \cdots,\ A^{n-1}B,\ D,\ AD,\ \cdots,\ A^{n-1}D) = n\,. \tag{8}$$

We also introduce following lemmas.

Lemma 2.4. (See [15, Theorem 2.3.1].) Provided impulsive system

$$\begin{aligned} &\dot{x}(t) = A(t)x(t) + B(t)u(t),\ t \neq t_i \\ &x(t_i^+) = x(t_i^-) + D(t_i)u_i,\ t = t_i \end{aligned} \tag{9}$$

where $A(t),\ B(t),\ D(t),\ t \in [t_0,\ t^*]$ are $n\times n,\ n\times r,\ n\times r$ continuous matrix functions respectively, then the corresponding linear system without impulses and with ordinary control, or, without ordinary control and with impulses, is completely controllable, if and only if

1) the $n\times r$ matrix function $X(t)X^{-1}(t_0)B(t)$ are linearly independent on $[t_0,\ t^*]$,

or,

2) there exist time instants $t_i \in [t_0,\ t^*],\ i = 1,\ 2,\ \cdots,\ l,\ l \in N,\ t_1 < t_2 < \cdots < t_l$ and vectors $u_i \in R^r,\ i = 1,\ 2,\ \cdots,\ l$, such that

$$rank(X^{-1}(t_1)D(t_1),\ X^{-1}(t_2)D(t_2),\ \cdots,\ X^{-1}(t_l)D(t_l)) = n \tag{10}$$

Lemma 2.5. (See [6, Theorem 3].) Impulsive system (4) is completely controllable, if and only if pair $\{A,\ B\}$ is controllable.

3. Relative Controllability of a Linear Time-Varying Impulsive System

Now we first are ready to state a necessary and sufficient condition as a criterion to guarantee linear time-varying impulsive system (1) is relatively controllable.

Theorem 3.1. System (1) is relatively controllable, if and only if there exists a certain family of set

$$T_{sp} := \{T_B(k),\ k \in K_{sp};\ T_D(p),\ p \in P_{sp}\}$$

such that

$$\begin{gathered} T_B(k) \subseteq [t_0,\ t^*],\ k \in K_{sp} \subseteq K;\ T_D(p) \subseteq [t_0,\ t^*],\ p \in P_{sp} \subseteq P \\ \sum_{k\in K_{sp}} |T_B(k)| + \sum_{p\in P_{sp}} |T_D(p)| = m \\ rank(HX(t^*,\ t)b_k(t),\ t \in T_B(k),\ k \in K_{sp}; \\ HX(t^*,\ t)d_p(t),\ t \in T_D(p),\ p \in P_{sp}) = rankH \end{gathered} \tag{11}$$

hold, where in left side of last equality the expression with bracket denotes the matrix taking columns

$$HX(t^*,\ t)b_k(t),\ t \in T_B(k),\ k \in K_{sp};\ HX(t^*,\ t)d_p(t),\ t \in T_D(p),\ p \in P_{sp}\,,$$

and $|T_B(k)|,\ |T_D(p)|$ - numbers of elements of given finite sets $T_B(k),\ T_D(p)$ respectively.

Proof. Necessity. We divide the proof of necessity into two steps.

Step 1. We claim that if impulsive system (1) is relatively controllable, then for any vector $y \in R^m$ with $\|y\| = 1$,

$$y' HX(t^*,\ t)B(t) \neq 0,\ t \in [t_0,\ t^*] \ \text{ or } \ y' HX(t^*,\ t)D(t) \neq 0,\ t \in [t_0,\ t^*] \tag{12}$$

holds, where y' denotes the transpose of y. By contradiction we are going to prove our claim. In fact, in opposition to the claim, let's assume that for a certain vector $y_* \in R^m$ with $\|y_*\| = 1$

$$y_*' HX(t^*, t)B(t) \equiv 0,\ t \in [t_0, t^*] \ \text{ and } \ y_*' HX(t^*, t)D(t) \equiv 0,\ t \in [t_0, t^*] \tag{13}$$

simultaneously hold. Now we choose a vector $g_* \in R^m$ satisfying the relation following

$$y_*'(g_* - HX(t^*, t_0)x_0) \neq 0\ . \tag{14}$$

Since if otherwise, then equality $y_* = 0$ should be resulted, that is contradictory with $\|y_*\| = 1$, it is guaranteed that the vector $g_* \in R^m$ mentioned above always exists. On one hand, since impulsive system (1) is relatively controllable, we know from definition 2.1 that for any initial state $x_0 \in R^n$ and chosen $g_* \in R^m$, there is a suitable impulsive control

$$\{u(\cdot),\ w\} := \{u(t),\ \ t \in [t_0,\ t^*],\ \{(t_i,\ v_i),\ \ i = 1,\ 2,\ \cdots, l\}\}$$

with which corresponding trajectory $x(t),\ t \in [t_0, t^*]$ satisfies terminal condition $Hx(t^*) = g_*$. Then we have by formula (3) following

$$\int_{t_0}^{t^*} HX(t^*, t)B(t)u(t)dt + \sum_{i=1}^{l} HX(t^*, t_i)D(t_i)v_i \ = g_* - HX(t^*, t_0)x_0\ .$$

Therefore, an equality is obtained as

$$\int_{t_0}^{t^*} y_*' HX(t^*, t)B(t)u(t)dt + \sum_{j=1}^{l} y_*' HX(t^*, t_j)D(t_j)v_j = y_*'(g_* - HX(t^*, t_0)x_0)\ ,$$

but this is a contradiction, since left side of the equality above has to be 0 by (13), while right side not to be 0 by (14). Thus our claim has been proved.

Step 2. We claim that if for any vector $y \in R^m$ with $\|y\| = 1$, (12) holds, then there exists a certain family of set T_{sp} satisfying (11).

We know from (12) that for arbitrarily given vector $y_0 \in R^m$ with $\|y_0\| = 1$

$$y_0' HX(t^*, t)B(t) \neq 0,\ t \in [t_0, t^*] \tag{15}$$

or

$$y_0' HX(t^*, t)D(t) \neq 0,\ t \in [t_0, t^*] \tag{16}$$

holds, therefore, first of all, suppose that (15) is true. In the case of (15), there is at least one index $k_1 \in K$ which

$$y_0' HX(t^*, t)b_{k_1}(t) \neq 0,\ t \in [t_0, t^*]$$

holds for, where $b_{k_1}(t)$ is k_1-th column of matrix $B(t)$. It is clear that

$$\sup_{t \in [t_0,\ t^*]} \left| y_0' HX(t^*, t)b_{k_1}(t) \right| > 0\ ,$$

because matrix function $B(t),\ t \in [t_0, t^*]$ is left and piecewise continuous. Hence for a suitable $\tau_{k_1}^1 \in [t_0, t^*]$

$$\left| y_0' HX(t^*, \tau_{k_1}^1)b_{k_1}(\tau_{k_1}^1) \right| > 0$$

holds. Besides we have

$$rank(HX(t^*,\ \tau_{k_1}^1)b_{k_1}(\tau_{k_1}^1)) = 1\,.$$

Now, introducing following notations

$$K^1 := \{k_1\},\ T_B^1(k) := \{\tau_k^1\},\ k \in K^1;$$
$$P^1 := \varphi,\ T_D^1(p) := \varphi,\ p \in P^1,$$

where φ denotes the empty set, we construct sets as

$$K^1 \subseteq K,\ T_B^1(k) \subseteq [t_0,\ t^*],\ k \in K^1;\ P^1 \subseteq P,\ T_D^1(p) \subseteq [t_0,\ t^*],\ p \in P^1$$

satisfying the relation

$$\begin{aligned} &rank(HX(t^*,\ t)b_k(t),\ t \in T_B^1(k),\ k \in K^1; \\ &HX(t^*,\ t)d_p(t),\ t \in T_D^1(p),\ p \in P^1) = 1 \end{aligned} \tag{17}$$

Second, according to same reasoning as the case (15), under the condition (16), by correcting notations obtained above as

$$K^1 := \varphi,\ T_B^1(k) := \varphi,\ k \in K^1;\ P^1 := \{p_1\} \subseteq P,\ T_D^1(p) := \{\tau_p^1\},\ p \in P^1,$$

we also obtain the same rank result just as (17).

We now suppose that we have already had a family of set

$$\{T_B^S(k),\ k \in K^S,\ T_D^S(p),\ p \in P^S\}$$

satisfying conditions as

$$\begin{gathered} T_B^S(k) \subseteq [t_0,\ t^*],\ k \in K^S \subseteq K;\ T_D^S(p) \subseteq [t_0,\ t^*],\ p \in P^S \subseteq P \\ \sum_{k \in K^S} \left|T_B^S(k)\right| + \sum_{p \in P^S} \left|T_D^S(p)\right| = s \\ rank(HX(t^*,\ t)b_k(t),\ t \in T_B^S(k),\ k \in K^S; \\ HX(t^*,\ t)d_p(t),\ t \in T_D^S,\ p \in P^S) = s \end{gathered} \tag{18}$$

Two cases are possible. In the first case as $s = m$, by defining as

$$T_B(k) := T_B^S(k),\ k \in K_{sp};\ P_{sp} := P^S,\ T(p) := T_D^S(p),\ p \in P_{sp}$$

we know that (18) exactly coincides with (11). So we consider the second case as $s < m$. Then, it is clear that a vector $y_s \in R^m$ with $\|y_s\| = 1$ exists which

$$\begin{aligned} &y_s^1 HX(t^*,\ t)b_k(t) = 0,\ t \in T_B^s(k),\ k \in K^s \\ &y_s^1 HX(t^*,\ t)b_p(t) = 0,\ t \in T_D^s(p),\ p \in P^s \end{aligned} \tag{19}$$

hold for. Rewriting (12) for $y = y_s$, we have following:

$$y_s' HX(t^*,\ t)B(t) \neq 0,\ t \in [t_0,\ t^*] \tag{20}$$

or

$$y_s' HX(t^*,\ t)D(t) \neq 0,\ t \in [t_0,\ t^*] \tag{21}$$

In the case (20) there exists a certain $k_{s+1} \in K$ which

$$y_s' HX(t^*,\ t)b_{k_{s+1}}(t) \neq 0,\ t \in [t_0,\ t^*]$$

holds for. Then we have

$$\sup_{t\in[t_0,\ t^*]} \left| y_s' HX(t^*,\ t) b_{k_{s+1}}(t) \right| > 0$$

because of left and piecewise continuity of function

$$y_s' HX(t^*,\ t) b_{k_{s+1}}(t),\ t\in[t_0,\ t^*].$$

This allows us to take a certain element $\tau_{k_{s+1}}^{s+1} \in [t_0,\ t^*]$ satisfying inequality

$$\left| y_s' HX(t^*,\ \tau_{k_{s+1}}^{s+1}) b_{k_{s+1}}(\tau_{k_{s+1}}^{s+1}) \right| > 0, \tag{22}$$

according to conditions below:

1) if $k_{s+1}\in K^s$, then $\tau_{k_{s+1}}^{s+1}\in[t_0,\ t^*]\setminus T^s(k_{s+1})$

2) if $k_{s+1}\notin K^s$, then $\tau_{k_{s+1}}^{s+1}\in[t_0,\ t^*]$.

Next, we construct sets $K^{s+1},\ T_B^{s+1},\ k\in K^{s+1};\ P^{s+1},\ T_D^{s+1}(p),\ p\in P^{s+1}$ as follows,

i). if $k_{s+1}\in K^s$, then

$$\begin{aligned} &K^{s+1} := K^s,\ T_B^{s+1}(k) := \begin{cases} T_B^s(k),\ k\neq k_{s+1} \\ T_B^s(k)\cup\left\{\tau_k^{s+1}\right\},\ k=k_{s+1},\ k\in K^{s+1} \end{cases} \\ &P^{s+1} := P^s,\ T_D^{s+1}(p) := T_D^s(p),\ p\in P^{s+1} \end{aligned} \tag{23}$$

ii). if $k_{s+1}\notin K^s$, then

$$\begin{aligned} &K^{s+1} := K^s\cup\left\{k_{s+1}\right\},\ T_B^{s+1}(k) := \begin{cases} T_B^s(k),\ k\neq k_{s+1} \\ \left\{\tau_{k_{s+1}}^{s+1}\right\},\ k=k_{s+1},\ k\in K^{s+1} \end{cases} \\ &P^{s+1} := P^s,\ T_D^{s+1}(p) := T_D^s(p),\ p\in P^{s+1} \end{aligned} \tag{24}$$

We now need to check the validity of following rank equality

$$\begin{aligned} &rank\ (HX(t^*,\ t)b_k(t),\ t\in T_B^{s+1}(k),\ k\in K^{s+1}; \\ &HX(t^*,\ t)d_p(t),\ t\in T_D^{s+1}(p),\ p\in P^{s+1}) = s+1 \end{aligned} \tag{25}$$

Let’s say no, then columns of the matrix in left side of (25) are linearly dependent. Hence, in the case (23) there are real numbers

$$\begin{gathered} \alpha_k^t,\ t\in T_B^{s+1}(k),\ k\in K^{s+1},\ k\neq k_{s+1},\ \alpha_{k_{s+1}}^t,\ t\in T_B^s(k_{s+1}); \\ \beta_p^t,\ t\in T_D^{s+1}(p),\ p\in P^{s+1} \end{gathered}$$

satisfying following equality

$$HX(t^*,\ \tau_{k_{s+1}}^{s+1}) b_{k_{s+1}}(\tau_{k_{s+1}}^{s+1}) = \sum_{k\in K^{s+1}} \Big(\sum_{\substack{t\in T_B^{s+1}(k) \\ k\neq k_{s+1}}} \alpha_k^t HX(t^*,\ t) b_k(t) + \sum_{t\in T_B^s(k_{s+1})} \alpha_{k_{s+1}}^t HX(t^*,\ t) b_{k_{s+1}}(t) \Big) + \sum_{p\in P^{s+1}} \sum_{t\in T_D^{s+1}} \beta_p^t HX(t^*,\ t) d_p(t),$$

and in the case (24) there are real numbers

$$\alpha_k^t,\ t\in T_B^s(k),\ k\in K^s;\ \beta_p^t,\ t\in T_D^{s+1}(p),\ p\in P^{s+1}$$

such as equality

$$HX(t^*,\ \tau_{k_{s+1}}^{s+1})b_{k_{s+1}}(\tau_{k_{s+1}}^{s+1})=\sum_{k\in K^s}\sum_{t\in T_B^s(k)}\alpha_k^t HX(t^*,\ t)b_k(t)+\sum_{p\in P^{s+1}}\sum_{t\in T_D^{s+1}(p)}\beta_p^t HX(t^*,\ t)d_p(t)$$

holds. Thus we have

$$y_s^{'}X(t^*,\ \tau_{k_{s+1}}^{s+1})b_{k_{s+1}}(\tau_{k_{s+1}}^{s+1})=\sum_{k\in K^{s+1}}\left(\sum_{\substack{t\in T_B^{s+1}(k)\\ k\neq k_{s+1}}}\alpha_k^t y_s^{'}HX(t^*,\ t)b_k(t)+\sum_{t\in T_B^s(k_{s+1})}\alpha_{k_{s+1}}^t y_s^{'}HX(t^*,\ t)b_{k_{s+1}}(t)\right)+\sum_{p\in P^{s+1}}\sum_{t\in T_D^{s+1}(p)}\beta_p^t y_s^{'}HX(t^*,\ t)d_p(t),$$

$$y_s^{'}HX(t^*,\ \tau_{k_{s+1}}^{s+1})b_{k_{s+1}}(\tau_{k_{s+1}}^{s+1})=\sum_{k\in K^s}\sum_{t\in T_B^s(k)}\alpha_k^t y_s^{'}HX(t^*,\ t)b_k(t)+\sum_{p\in P^{s+1}}\sum_{t\in T_D^{s+1}(p)}\beta_p^t y_s^{'}HX(t^*,\ t)d_p(t)$$

but both of right sides of two equalities take the value 0, because of (19). Therefore, we arrive at the result that for all possible cases

$$y_s^{'}HX(t^*,\ \tau_{k_{s+1}}^{s+1})b_{k_{s+1}}(\tau_{k_{s+1}}^{s+1})=0$$

which is in contradiction to (22). That is, it has been proved that (20) implies (25).

In similar manner we may construct sets

$$K^{s+1},\ T_B^{s+1}(k),\ k\in K^{s+1};\ P^{s+1},\ T_D^{s+1}(p),\ p\in P^{s+1}$$

such that (21) implies (25).

Continuously repeating the procedure mentioned above, just after m steps, we exactly have the family of set

$$T_{sp}=\{T_B(k),\ k\in K_{sp};\ T_D(p),\ p\in P_{sp}\}$$

satisfying (11) as follows:

$$K_{sp}:=K^m,\ T_B(k):=T_B^m(k),\ k\in K^m;$$
$$P_{sp}:=P^m,\ T_D(p):=T_D^m(p),\ p\in P^m.$$

Sufficiency. Suppose that there exists a certain family of set

$$T_{sp}=\{T_B(k),\ k\in K_{sp};\ T_D(p),\ p\in P_{sp}\}$$

such that the relations (11) hold.

To start with, we need to emphasize that without loss of generality, we may any element $t\in T_B(k)$ for all $k\in K_{sp}$ in the (11) to be a continuous point of the vector function $HX(t^*,\ t)b_k(t),\ t\in[t_0,\ t^*]$ because of left and piecewise continuity of the vector function itself.

Now, for arbitrary initial state $x_0\in R^n$ and vector $g\in R^m$, let's examine a construction of impulsive control $\{u(\cdot),\ w\}$ with which the corresponding trajectory $x(t),\ t\in[t_0,\ t^*]$ satisfies terminal state condition $HX(t^*)=g$. For that purpose, we first make components of vector function $u(t),\ t\in[t_0,\ t^*]$ as follows:

$$\begin{gathered}u_k(t):=\begin{cases}u_k^{\tau(k)}, & when\ \ t\in[\tau(k),\ \tau(k)+\varepsilon(\tau(k))),\ \tau(k)\in T_B(k)\\ 0, & when\ \ t\notin\underset{\tau(k)\in T_B(k)}{U}[\tau(k),\ \tau(k)+\varepsilon(\tau(k)))\end{cases},\\ t\in[t_0,\ t^*],\ k\in K_{sp};\\ u_k(t)\equiv 0,\ \ t\in[t_0,\ t^*],\ \ k\in K\backslash K_{sp}\end{gathered}\tag{26}$$

where $u_k(t)$ is k–th component of the vector $u(t)\in R^r$, $u_k^{\tau(k)}$-real number valued parameter and $\varepsilon(\tau(k))$- a sufficiently small positive real number. Second, taking in mind the order for elements of the set $\underset{p\in P_{sp}}{U}T_D(p),\ p\in P_{sp}$ as $t_1<t_2<\cdots<t_i$, we define following:

$$\begin{aligned} &v_t^p := c_t^p, \ \ t \in T_D(p), \ \ p \in P_{sp} \\ &v_t^p := 0, \ \ t \in \bigcup_{s \in P_{sp}} T_D(s) \backslash T_D(p), \ \ p \in P_{sp} \\ &v_t^p := 0, \ \ t \in \bigcup_{s \in P_{sp}} T_D(s), \ \ p \in P \backslash P_{sp} \end{aligned} \tag{27}$$

where v_t^p is p-th component of the vector $v_t \in P^q$, c_t^p-real number valued parameter. It is obvious that r vector function $u(t), \ t \in [t_0, \ t^*]$ generated by the formula (26) has piecewise continuity property, when under sufficiently small positive numbers $\varepsilon(\tau(k)), \ \tau(k) \in T_B(k), \ k \in K_{sp}$ concrete values of parameters $u_k^{\tau(k)}, \ \tau(k) \in T_B(k), \ k \in K_{sp}$ are taken. In addition, we have the impulsive action

$$w = \{(t, v_t), t \in \bigcup_{s \in P_{sp}} T_D(s)\}$$

which generated by the formula (27) under concrete values of parameters $c_t^p, \ t \in \bigcup_{s \in P_{sp}} T_D(s), \ p \in P$. Doubtlessly, the pair $\{u(\cdot), \ w\}$ consisting of the r vector function $u(t), \ t \in [t_0, \ t^*]$ and the impulsive action w which is constructed in this manner is exactly an impulsive control. Furthermore, it is desirable that with the impulsive control $\{u(\cdot), \ w\}$, corresponding trajectory $x(t), \ t \in [t_0, \ t^*]$ satisfies the terminal condition $HX(t^*) = g$. So according to formulas (26) and (27), we have following:

$$\begin{aligned} \int_{t_0}^{t^*} HX(t^*, \ t)B(t)u(t)dt + \sum_{t \in \bigcup_{s \in P_{sp}} T_D(s)} HX(t^*, \ t)D(t)v_t &= \sum_{k \in K} \int_{t_0}^{t^*} HX(t^*, \ t)b_k(t)u_k(t)dt + \sum_{t \in \bigcup_{s \in P_{sp}} T_D(s)} \sum_{p \in P} HX(t^*, \ t)d_p(t)v_t^p \\ &= \sum_{k \in K} \int_{t_0}^{t^*} HX(t^*, \ t)b_k(t)u_k(t)dt + \sum_{p \in P} \sum_{t \in \bigcup_{s \in P_{sp}} T_D(s)} HX(t^*, \ t)d_p(t)v_t^p \\ &= \sum_{k \in K_{sp}} \int_{t_0}^{t^*} HX(t^*, \ t)b_k(t)u_k(t)dt + \sum_{k \in K \backslash K_{sp}} \int_{t_0}^{t^*} HX(t^*, \ t)b_k(t)u_k(t)dt \\ &+ \sum_{p \in P_{sp}} \sum_{t \in \bigcup_{s \in P_{sp}} T_D(s)} HX(t^*, \ t)d_p(t)v_t^p + \sum_{p \in P \backslash P_{sp}} \sum_{t \in \bigcup_{s \in P_{sp}} T_D(s)} HX(t^*, \ t)d_p(t)v_t^p \\ &= \sum_{k \in K_{sp}} \int_{t_0}^{t^*} HX(t^*, \ t)b_k(t)u_k(t)dt + \sum_{p \in P_{sp}} \sum_{t \in \bigcup_{s \in P_{sp}} T_D(s)} HX(t^*, \ t)d_p(t)v_t^p \\ &= \sum_{k \in K_{sp}} \int_{t_0}^{t^*} HX(t^*, \ t)b_k(t)u_k(t)dt \\ &+ \sum_{p \in P_{sp}} \left(\sum_{t \in T_D(p)} HX(t^*, \ t)d_p(t)v_t^p + \sum_{t \in \bigcup_{s \in P_{sp}} T_D(s) \backslash T_D(p)} HX(t^*, \ t)d_p(t)v_t^p \right) \\ &= \sum_{k \in K_{sp}} \sum_{\tau(k) \in T_B(k)} \left(\int_{\tau(k)}^{\tau(k)+\varepsilon(\tau(k))} HX(t^*, \ t)b_k(t)dt \right) u_k^{\tau(k)} + \sum_{p \in P_{sp}} \sum_{t \in T_D(p)} HX(t^*, \ t)d_p(t)v_t^p. \end{aligned}$$

Eventually we may write terminal constraint as

$$\sum_{k \in K_{sp}} \sum_{\tau(p) \in T_B(k)} \left(\int_{\tau(k)}^{\tau(k)+\varepsilon(\tau(k))} HX(t^*, \ t)b_k(t)dt \right) u_k^{\tau(k)} + \sum_{p \in P_{sp}} \sum_{t \in T_D(p)} HX(t^*, \ t)d_p(t)v_t^p = g - HX(t^*, \ t_0)x_0. \tag{28}$$

Moreover, as we mentioned before, since vector function $HX(t^*, \ t)b_k(t), \ t \in [t_0, \ t^*]$ is continuous in the point $\tau(k) \in T_B(k), \ k \in K_{sp}$, we have following by using of Taylor's formula

$$\int_{\tau(k)}^{\tau(k)+\varepsilon(\tau(k))} HX(t^*,\ t)b_k(t)dt = \varepsilon(\tau(k))HX(t^*,\ \tau(k))b_k(\tau(k)) + \varepsilon(\tau(k))\beta(\tau(k)),$$
$$\lim_{\varepsilon(\tau(k))\downarrow 0} \beta(\varepsilon(\tau(k))) = 0,\ \tau(k)\in T_B(k),\ k\in K_{sp}. \tag{29}$$

With regard to (29), we obtain the other representation of (28) as

$$\begin{aligned}&(\varepsilon(\tau(k))\left(HX(t^*,\ t)b_k(t)+\beta(\tau(k))\right),\ \tau(k)\in T_B(k),\ k\in K_{sp};\\ &HX(t^*,\ t)d_p(t),\ t\in T_D(p),\ p\in P_{sp})(u_k^{\tau(k)},\ \tau(k)\in T_B(k),\ k\in K_{sp};\\ &v_t^p,\ t\in T_D(p),\ p\in P_{sp}) = g - HX(t^*,\ t_0)x_0,\end{aligned} \tag{30}$$

where to simplify the notation and for our convenience, in the left side of the equality above we denote the $m\times m$ matrix with columns

$$\begin{aligned}&\varepsilon(\tau(k))\left(HX(t^*,\ \tau_k)b_k(\tau_k)+\beta(\tau(k))\right),\ \tau(k)\in T_B(k),\ k\in K_{sp};\\ &HX(t^*,\ t)d_p(t),\ t\in T_D(p) p\in P_{sp}\end{aligned}$$

as the expression with first parenthesis and the m vector with components

$$u_k^{\tau(k)},\ \tau(k)\in T_B(k),\ k\in K_{sp};\ v_t^p,\ t\in T_D(p),\ p\in P_{sp}$$

as the expression with second parenthesis. Based on the discussed results above and the assumption (11) for sufficiency of our theorem, we arrive at the fact that there is positive ε such that for any pair

$$\varepsilon(\tau(k)),\ \tau(k)\in T_B(k),\ k\in K_{sp}$$

satisfying conditions

$$0<\varepsilon(\tau(k))<\varepsilon,\ \tau(k)\in T_B(k),\ k\in K_{sp}$$

the coefficient matrix in left side of (30) is non-singular. Furthermore, this implies, in particular, that under such pair $\varepsilon(\tau(k)),\ \tau(k)\in T_B(k),\ k\in K_{sp}$, the equation (30) has the unique solution as

$$\begin{aligned}&(u_k^{\tau(k)},\ \tau(k)\in T_B(k),\ k\in K_{sp};\ v_t^p,\ t\in T_D(p),\ p\in P_{sp}) =\\ &= (\varepsilon(\tau(k))\left(HX(t^*,\ t)b_k(t)+\beta(\tau(k))\right),\ \tau(k)\in T_B(k),\ k\in K_{sp};\\ &HX(t^*,\ t)d_p(t),\ t\in T_D(p) p\in P_{sp})^{-1}(g-HX(t^*,\ t_0)x_0).\end{aligned} \tag{31}$$

Thus, it has been proved that with impulsive control $\{u(\cdot),\ w\}$ defined by expressions (26), (27) with the values of parameters as the solution (31), the corresponding trajectory satisfies the terminal condition $HX(t^*)=g$. Moreover, this also means that the system (1) is relatively controllable. The proof is finished.

Next, for linear time-varying impulsive system (1) we easily have the following alternative criterion to check relative controllability, which soon follows as a sequence of the theorem 3.1 above.

Theorem 3.2. The linear time-varying impulsive system (1) is relatively controllable if and only if for any vector $y\in R^m$ with $\|y\|=1$ holds.

$$y' HX(t^*,\ t)B(t)\neq 0,\ t\in[t_0,\ t^*]\ \text{ or }\ y' HX(t^*,\ t)D(t)\neq 0,\ t\in[t_0,\ t^*] \tag{32}$$

Proof. Necessity. In the step 1 for the proof of necessity of the theorem 3.1, it has already been verified that if the system (1) is relatively controllable, then for any vector $y\in R^m$ with $\|y\|=1$, (32) holds.

Sufficiency. We need to show that if for any vector $y\in R^m$ with $\|y\|=1$, (32) holds, then the system (1) is relatively controllable. In the first place we can remember that in the step 2 for the proof of necessity of the theorem 3.1, one has already obtained the sequence such that if for any vector $y\in R^m$ with $\|y\|=1$, (32) holds, then there exists a certain family of set

$$T_{sp}=\{T_B(k),\ k\in K_{sp};\ T_D(p),\ p\in P_{sp}\}$$

satisfying (11). Hence, from this fact and sufficient condition of the theorem 3.1 we arrive at relative controllability of linear time-varying impulsive system (1). This completes our proof.

Theorem 3.3. Given a matrix $H\in R^{m\times n}$ with $rank\ H=m$, then the following two statements are equivalent:

(a) the impulsive system (1) is relatively controllable;

(b) the impulsive system (1) is relatively null controllable.

Proof. $(a)\Rightarrow(b)$. This implication is obvious from both of definitions of relative and relative null controllabilities.

$(b)\Rightarrow(a)$. Assume that the system (1) is relatively null controllable. We now claim that for any $y\in R^m$ with $\|y\|=1$, the relation (12) holds. In fact, if otherwise, then for a certain vector $y_*\in R^m$ with $\|y_*\|=1$, (13) should hold. In this case, it is easy to verify that there exists the vector $x_0^*\in R^n$ such that

$$y_*^{'}HX(t^*,\ t_0)x_0^*\neq 0 \tag{33}$$

because of $rank\ H=m$ and non-singularity of the matrix $X(t^*,\ t_0)$.

We also know from the definition for concept of relative null controllability that there exists at least one impulsive control

$$\{u(t),\ t\in[t_0,\ t^*],\ \{\{t_i,\ v_i\},\ i=1,\ 2,\ \cdots,\ l\}\}$$

such that the relation

$$HX(t^*,\ t_0)x_0^*+\int_{t_0}^{t^*}HX(t^*,\ t_i)B(t)u(t)dt+\sum_{i=1}^{l}HX(t^*,\ t_i)D(t_i)v_i=0$$

holds. This implies that the equation

$$\int_{t_0}^{t^*}y_*^{'}HX(t^*,\ t)B(t)u(t)dt+\sum_{i=1}^{l}y_*^{'}HX(t^*,\ t_i)D(t_i)v_i=-y_*^{'}HX(t^*,\ t_0)x_0^*$$

holds, but this relation yields a contradiction because of (13) and (33).

Thus, our claim such that for any $y\in R^m$ with $\|y\|=1$ (12) is true, has been proved. Consequently, from theorem 3.2, we get relative controllability of impulsive system (1). This completes our proof.

Remark 3.1. Eventually, by the theorems 3.1, 3.2, 3.3, it is verified that the following four propositions are equivalent one another:

Proposition 1. The linear time-varying impulsive system (1) is relatively controllable;

Proposition 2. The linear time-varying impulsive system (1) is relatively null controllable;

Proposition 3. There exists a certain family of set

$$\{T_B(k),\ k\in K_{sp};\ T_D(p),\ p\in P_{sp}\}$$

satisfying the relation

$$\begin{gathered}T_B(k)\subseteq[t_0,\ t^*],\ k\in K_{sp}\subseteq K;\ T_D(p)\subseteq[t_0,\ t^*],\ p\in P_{sp}\subseteq P\\ \sum_{k\in K_{sp}}|T_B(k)|+\sum_{p\in P_{sp}}|T_D(p)|=m\\ rank\ (HX(t^*,t)b_k(t),\ t\in T_B(k),k\in K_{sp};\\ HX(t^*,t)d_p(t),\ t\in T_D(p),p\in P_{sp})=rank\ H.\end{gathered} \tag{34}$$

Proposition 4. For any vector $y\in R^m$ with $\|y\|=1$

$$y^{'}HX(t^*,\ t)B(t)\neq 0,\ t\in[t_0,\ t^*]\ \text{ or }\ y^{'}HX(t^*,\ t)D(t)\neq 0,\ t\in[t_0,\ t^*] \tag{35}$$

holds.

In particular, we easily get the following corollary obtained from the above propositions when $H=I$.

Corollary 3.4. In the linear time-varying impulsive system (1) the following four statements are equivalent:

(a) The system (1) is completely controllable;

(b) The system (1) is null controllable;

(c) There exists a certain family of set

$$\{T_B(k) \subseteq [t_0,\ t^*],\ k \in K_{sp} \subseteq K;\ T_D(p) \subseteq [t_0,\ t^*],\ p \in P_{sp} \subseteq P\}$$

such that

$$\begin{aligned}&\sum_{k \in K_{sp}} |T_B(k)| + \sum_{p \in P_{sp}} |T_D(p)| = n \\ &rank\ (X(t^*,\ t)b_k(t),\ t \in T_B(k), k \in K_{sp}; \\ &X(t^*,\ t)d_p(t),\ t \in T_D(p), p \in P_{sp}) = n\end{aligned} \tag{36}$$

holds.

(d) For any vector $y \in R^n$ with $\|y\| = 1$

$$y' X(t^*,\ t)B(t) \neq 0,\ t \in [t_0,\ t^*] \ \text{ or } \ y' X(t^*,\ t)D(t) \neq 0,\ t \in [t_0,\ t^*] \tag{37}$$

holds.

Remark 3.2. As already mentioned in the work of George R. K. et al. [2] (2000), generally speaking, for impulsive systems complete and null controllabilities are not equivalent, but fortunately, we have clarified that for the linear time-varying impulsive system (1) these two properties are just equivalent as in the corollary 3.4.

Remark 3.3. As compared with Lemma 2.4, we easily know that Corollary 3.4 implies that Lemma 2.4 is true.

4. Relative Controllability of Linear Time Invariant Impulsive System

In this section we are going to consider time-invariant impulsive system (1) when $A(t) = A,\ B(t) = B,\ D(t) = D$ where $A,\ B,\ D$ are $n \times n,\ n \times r,\ n \times q$ constant matrices respectively. For the time-invariant impulsive system we have further concise results.

Theorem 4.1. The linear time-invariant impulsive system (1) with $A(t) = A,\ B(t) = B,\ D(t) = D$ is relatively controllable, if and only if

$$rank\ (HB,\ HAB,\ \cdots,\ HA^{n-1}B;\ HD,\ HAD,\ \cdots,\ HA^{n-1}D) = rank\ H \tag{38}$$

Proof. Necessity. To prove the necessity by contradiction, we assume that

$$rank\ (HB,\ HAB,\ \cdots,\ HA^{n-1}B;\ HD,\ HAD,\ \cdots,\ HA^{n-1}D) < m = rank\ H \tag{39}$$

This implies that there exists at least one vector $\bar{y} \in R^m$ with $\|\bar{y}\| = 1$ such that

$$\bar{y}'\ (HB,\ HAB,\ \cdots,\ HA^{n-1}B;\ HD,\ HAD,\ \cdots,\ HA^{n-1}D) = 0 \tag{40}$$

We introduce following notations for some functions

$$\begin{aligned}\xi_k(t,\ y) &:= y'HX(t^*,\ t)b_k,\ \ t \in [t_0,\ t^*],\ k \in K \\ \eta_p(t,\ y) &:= y'HX(t^*,\ t)d_p,\ \ t \in [t_0,\ t^*],\ p \in P\end{aligned} \tag{41}$$

Let

$$\Lambda(\lambda) := \lambda^n + \sum_{i=0}^{n-1} a_{n-i} \lambda^i$$

be the characteristic polynomial of A. By Cayley-Hamilton theorem we know that

$$A^n + \sum_{i=0}^{n-1} a_{n-i} A^i = 0 ,$$

which we have following from:

$$\begin{aligned} &\bar{y}' HX(t^*,t)A^n b_k + \sum_{i=0}^{n-1} a_{n-i} \bar{y}' HX(t^*,t)A^i b_k = 0, k \in K \\ &\bar{y}' HX(t^*,t)A^n d_p + \sum_{i=0}^{n-1} a_{n-i} \bar{y}' HX(t^*,t)A^i d_p = 0,\ p \in P \end{aligned} \tag{42}$$

On one hand, sequentially differentiated functions $\xi_k(t,\bar{y}),\ k \in K$ and $\eta_p(t,\bar{y}),\ p \in P$ with respect to variable t, we have

$$\begin{aligned} &\xi_k^{(i)}(t,\bar{y}) = (-1)^i \bar{y}' HX(t^*,\ t)A^i b_k,\ \ i = 0,\ 1,\ 2,\ \cdots,\ n,\ k \in K \\ &\eta_k^{(i)}(t,\bar{y}) = (-1)^i \bar{y}' HX(t^*,\ t)A^i d_p,\ i = 0,\ 1,\ 2,\ \cdots,\ n,\ p \in P \end{aligned} \tag{43}$$

where $\xi_k^{(i)}(t,\bar{y})$, $\eta_k^{(i)}(t,\bar{y})$ denote i-th order derivatives of corresponding functions $\xi_k(t,\bar{y})$, $\eta_p(t,\bar{y})$ respectively. In particular, when $t = t^*$, we know that

$$\begin{aligned} &\xi_k^{(i)}(t^*,\bar{y}) = (-1)^i \bar{y}' HA^i b_k,\ i = 0,\ 1,\ 2,\ \cdots,\ n-1,\ k \in K \\ &\eta_k^{(i)}(t^*,\bar{y}) = (-1)^i \bar{y}' HA^i d_p,\ i = 0,\ 1,\ 2,\ \cdots,\ n-1,\ p \in P \end{aligned} \tag{44}$$

Then, with regard to (43) in the equations (42), we also have following:

$$\begin{aligned} &(-1)^n \xi_k^{(i)}(t,\bar{y}) + \sum_{i=0}^{n-1} (-1)^i a_{n-i} \xi_k^{(i)}(t,\bar{y}) = 0,\ k \in K \\ &(-1)^n \eta_p^{(i)}(t,\bar{y}) + \sum_{i=0}^{n-1} (-1)^i a_{n-i} \eta_p^{(i)}(t,\bar{y}) = 0,\ p \in P \end{aligned} \tag{45}$$

Hence, it is clear by the expressions (44), (45) that functions $\xi_k(t,\bar{y}),\ k \in K,\ \eta_p(t,\bar{y}),\ p \in P$ are solutions of the differential equation

$$(-1)^n z^{(n)}(t) + \sum_{i=0}^{n-1} (-1)^i a_{n-i} z^{(i)}(t) = 0 \tag{46}$$

correspondingly satisfying initial conditions

$$\begin{aligned} &z^{(i)}(t^*) = \xi_k^{(i)}(t^*,\bar{y}) = 0,\ i = 0,\ 1,\ 2,\ \cdots,\ n-1,\ k \in K \\ &z^{(i)}(t^*) = \eta_p^{(i)}(t^*,\bar{y}) = 0,\ i = 0,\ 1,\ 2,\ \cdots,\ n-1,\ p \in P \end{aligned} \tag{47}$$

respectively, because of (40). Therefore, from conditions (47), and the fact such that a homogeneous differential equation with trivial initial condition have only trivial solution, we have

$$\begin{aligned} &\xi_k(t,\bar{y}) \equiv 0,\ t \in [t_0,\ t^*],\ k \in K, \\ &\eta_p(t,\bar{y}) \equiv 0,\ t \in [t_0,\ t^*],\ p \in P \end{aligned} \tag{48}$$

which immediately yields

$$\bar{y}' HX(t^*,t)B \equiv 0, t \in [t_0, t^*]$$

and

$$\bar{y}' HX(t^*,t)D \equiv 0, t \in [t_0, t^*] .$$

But this is in contradiction to the condition (32), where $A(t) = A,\ B(t) = B,\ D(t) = D,$ for relative controllability of the time-invariant impulsive system. Thus it is proved that (38) holds.

Sufficiency. Assume that (38) holds. Then equation

$$y'(HB,\ HAB,\ \cdots,\ HA^{n-1}B;\ HD,\ HAD,\ \cdots,\ HA^{n-1}D) = 0$$

has only trivial solution. Hence, for any $y \in R^m$ with $\|y\| = 1$ there exists at least one non-null from among the numbers

$$\begin{aligned} &y' HA^i b_k,\ i = 0,\ 1,\ 2,\ \cdots,\ n-1,\ k \in K \\ &y' HA^i d_p,\ i = 0,\ 1,\ 2,\ \cdots,\ n-1,\ p \in P. \end{aligned} \tag{49}$$

And we easily know that functions defined as in the (41)

$$\xi_k(t, y),\ k \in K,\ \eta_p(t, y),\ p \in P,$$

all are solutions of the differential equation (46) with initial conditions

$$\begin{aligned} &z^{(i)}(t^*) = \xi_k^{(i)}(t^*, y) = (-1)^i y' HA^i b_k,\ i = 0,\ 1,\ 2,\ \cdots,\ n-1,\ k \in K \\ &z^{(i)}(t^*) = \eta_p^{(i)}(t^*, y) = (-1)^i y' HA^i d_p,\ i = 0,\ 1,\ 2,\ \cdots,\ n-1,\ p \in P. \end{aligned}$$

Consequently, with the help of the conclusion above such that there exists at least one non-null from among the numbers of the (49), this immediately implies that for any $y \in R^m$ with $\|y\| = 1$

$$y' HX(t^*, t)B \neq 0,\ \ t \in [t_0,\ t^*]$$

or

$$y' HX(t^*, t)D \neq 0,\ \ t \in [t_0,\ t^*]$$

holds. In the light of Theorem 3.2, this so too shows us that the linear time-invariant impulsive system is relatively controllable. Thus our theorem is completely proved.

By virtue of Theorem 4.1 we straightforward obtain following corollary which gives a criterion for complete controllability of given linear time-invariant impulsive system.

Corollary 4.2. The linear time-invariant impulsive system (1) with $A(t) = A,\ B(t) = B,\ D(t) = D$ is completely controllable, if and only if

$$rank\ (B,\ AB,\ \cdots,\ A^{n-1}B;\ D,\ AD,\ \cdots,\ A^{n-1}D) = n. \tag{50}$$

Remark 4.1. Comparatively considering, we can easily verify that Corollary 4.2 also implies that Lemmas 2.2 and 2.5 are true. This shows us that the condition (50) is more general and new as compared with the condition (5) and the condition such that pair $\{A,\ B\}$ is controllable.

Equipped with the necessary and sufficient condition (50) above for complete controllability in the corollary 4.2, we can now clarify relationships between null controllabilities and complete controllabilities of various kinds for the linear time-invariant impulsive system (6) as follows.

Theorem 4.3. Given the linear time-invariant impulsive system (1) with $r = q,\ A(t) = A,\ B(t) = B,\ D(t) = D,$ that is, given impulsive system (6), then the following four statements are equivalent:

(a) The system is null controllable;

(b) The system is completely controllable;

(c) There exists at least one common sequence of time instants for impulsive actions, which the system is null controllable;

(d) There exists at least one common sequence of time instants for impulsive actions, which the system is completely controllable.

Proof. $(a) \Leftrightarrow (b)$. It is obvious by virtue of the corollary 3.4 that in particular, given $r = q,\ A(t) = A,\ B(t) = B,\ D(t) = D$, that statements (a) and (b) are equivalent.

$(b) \Leftrightarrow (c)$. First, assume that the system is completely controllable. Then we know from Theorem 3.1 that there exists at least one family of set

$$\left\{T_B(k),\ k\in K_{sp};\ \ T_D(p),\ p\in P_{sp}\right\}$$

such that

$$\begin{gathered} T_B(k)\subseteq[t_0,\ t^*],\ k\in K_{sp}\subseteq K,\ T_D(p)\subseteq[t_0,\ t^*],\ p\in P_{sp}\subseteq P \\ \sum_{k\in K_{sp}}\left|T_B(k)\right|+\sum_{p\in P_{sp}}\left|T_D(p)\right|=n \\ rank\ \left(X(t^*,\ t)b_k(t),\ t\in T_B(k),\ k\in K_{sp};\ X(t^*,\ t)d_p(t),\ t\in T_D(p),\ p\in P_{sp}\right)=n \end{gathered} \tag{51}$$

holds.

Next, by using of (51), according to the procedure as in the proof for sufficiency in Theorem 3.1, we verify that for any initial vector $x_0\in R^n$, if we concretely give suitable values of given parameters in (26) and (27), then with the impulsive control $\{u(\cdot),w\}$ consisting of both of the ordinary control $u(t),\ t\in[t_0,t^*]$ defined by (26) and the action of impulses w defined by (27) with common sequence of time instants

$$t_1<t_2<\cdots<t_l$$

where

$$\{\ t_1,\ t_2,\ \cdots,\ t_l\}=\bigcup_{p\in P_{sp}}T_D(p)\,,$$

the corresponding trajectory $x(t),\ t\in[t_0,\ t^*]$ satisfies terminal condition $x(t^*)=0$. In other words, the statement (b) implies that the statement (c) is true.

Conversely, assume that statement (c) is true. Then, it follows that (50) holds because of Lemma 2.3. Hence, by virtue of Corollary 4.2 we know that the system is completely controllable.

$(c)\Leftrightarrow(d)$ First, assume that statement (c) is true. Then, this implies that the system is completely controllable, because (b) and (c) are equivalent. Therefore, from theorem 3.1 we know that there exists at least one family of set

$$\left\{\ T_B(k),\ k\in K_{sp};\ T_D(p),\ p\in P_{sp}\right\}$$

such that (51) holds. Next, according to the procedure as in the proof of the implication $(b)\Rightarrow(c)$, in similar manner, we easily verify that statement (d) is true.

The converse implication such that statement (d) implies that statement (c) is true, is clear by the definitions of notions. Thus, our proof is finished.

5. Example

We now consider the following two-dimensional linear time-invariant impulsive system with one-dimensional ordinary control and one-dimensional action of impulses given as below.

$$\begin{gathered} \dot{x}_1(t)=x_1(t)+\frac{1}{2}x_2(t)+u(t) \\ \dot{x}_2(t)=x_1(t)+\frac{3}{2}x_2(t)+2u(t),\ \ t\neq t_i \\ x_1(t_i^+)=x_1(t_i^-)+2v_i \\ x_2(t_i^+)=x_2(t_i^-)+4v_i,\ \ t\neq t_i \\ x(t_0^+)=x_0,\ \ t_0=0,\ \ t^*=1 \end{gathered} \tag{52}$$

where

$$A=\begin{pmatrix}1 & \frac{1}{2}\\ 1 & \frac{3}{2}\end{pmatrix},\ B=\begin{pmatrix}1\\2\end{pmatrix},\ D=\begin{pmatrix}2\\4\end{pmatrix}.$$

First of all, we clarify that the impulsive system (52) isn't not only completely controllable but also null controllable. For this purpose, we choose the initial state and the desired terminal state vectors as follows:

$$x_0=\begin{pmatrix}0\\0\end{pmatrix},\ x_f=\begin{pmatrix}1\\3\end{pmatrix}.$$

The characteristic equation and a fundamental matrix for the corresponding homogeneous system is given by

$$2\lambda^2-5\lambda+2=0;$$

$$X(t)=\begin{pmatrix}e^{\frac{1}{2}t} & e^{2t}\\ -e^{\frac{1}{2}t} & 2e^{2t}\end{pmatrix}$$

where the eigenvalues and the eigenvectors as follows, respectively:

$$\lambda_1=\frac{1}{2},\ r_1=\begin{pmatrix}1\\-1\end{pmatrix};$$

$$\lambda_2=2,\ r_2=\begin{pmatrix}1\\2\end{pmatrix}.$$

Hence, the inverse matrix of $X(t)$ is

$$X^{-1}(t)=\begin{matrix}\frac{2}{3}e^{\frac{1}{2}t} & \frac{1}{3}e^{\frac{1}{2}t}\\ \frac{1}{3}e^{2t} & \frac{1}{3}e^{2t}\end{matrix}$$

And the state transition matrix associated with matrix A is

$$X(t,\ s)=\begin{pmatrix} \frac{2}{3}e^{\frac{1}{2}(t-s)}+\frac{1}{3}e^{2(t-s)} & -\frac{1}{3}e^{\frac{1}{2}(t-s)}+\frac{1}{3}e^{2(t-s)} \\ -\frac{2}{3}e^{\frac{1}{2}(t-s)}+\frac{2}{3}e^{2(t-s)} & \frac{1}{3}e^{\frac{1}{2}(t-s)}+\frac{2}{3}e^{2(t-s)} \end{pmatrix}.$$

Then, according to the formula (3), with the impulsive control

$$\{u(\cdot),\ \{(t_i,\ v_i),\ i=1,\ 2,\ \cdots,\ l\}\}$$

the corresponding terminal state of the trajectory has following type.

$$x(1)=\int_0^1 X(1,\ t)Bu(t)dt+\sum_{i=1}^{l}X(1,\ t_i)Dv_i,$$

where

$$X(1,\ t)=\begin{pmatrix} \frac{2}{3}e^{\frac{1}{2}(1-t)}+\frac{1}{3}e^{2(1-t)} & -\frac{1}{3}e^{\frac{1}{2}(1-t)}+\frac{1}{3}e^{2(1-t)} \\ -\frac{2}{3}e^{\frac{1}{2}(1-t)}+\frac{2}{3}e^{2(1-t)} & \frac{1}{3}e^{\frac{1}{2}(1-t)}+\frac{2}{3}e^{2(1-t)} \end{pmatrix}.$$

Therefore,

$$x(1)=\int_0^1\begin{pmatrix} e^{2(1-t)} \\ 2e^{2(1-t)}\end{pmatrix}u(t)dt+\sum_{i=1}^{l}\begin{pmatrix} 2e^{2(1-t_i)} \\ 4e^{2(1-t_i)}\end{pmatrix}v_i,$$

that is, we have

$$\begin{aligned} x_1(1)&=\int_0^1 e^{2(1-t)}u(t)dt+\sum_{i=1}^{l}2e^{2(1-t_i)}v_i \\ x_2(1)&=2\left(\int_0^1 e^{2(1-t)}u(t)dt+\sum_{i=1}^{l}2e^{2(1-t_i)}v_i\right)\end{aligned}.$$

Thus, we soon know that if there exists an impulsive control such that $x_1(1)=1$, then $x_2(1)=2$ needs to hold, that is $x_2(1)\neq 3$. Clearly, no impulsive control $\{u(\cdot),\ w\}$ will steer the initial vector $x_0=(0,\ 0)'$ to the terminal vector $x_f=(1,\ 2)'$.

This shows that the impulsive system (52) is not completely controllable that coincides with the statement of Corollary 4.2 because

$$rank\ (B,\ AB,\ D,\ AD)=rank\ \begin{pmatrix} 1 & 2 & 2 & 4 \\ 2 & 4 & 4 & 8\end{pmatrix}=1<2=n\ .$$

Besides, with the help of Theorem 4.3 we so too know that the system (52) is not null controllable.

Next, we can prove that the system (52) is relatively controllable with respect to the terminal constraint

$$x_1(1)+2x_2(1)=g \tag{53}$$

where

$$rank\ H=rank\ (1,\ 2)=1.$$

In fact, we have

$$rank\ (HB,\ HAB,\ HD,\ HAD)=rank\ (5,\ 10,\ 10,\ 20)=rank\ H\ .$$

Hence, by Theorem 4.1, the impulsive system (52) is relatively controllable with respect to the terminal state constraint (53). What this example does for us here is guaranteed that the concept of relative controllability with respect to right terminal state constraint is more general than those of complete and null controllabilities because for the system (1) complete or null controllabilities imply obviously that the system is relatively controllable.

6. Conclusion

In this paper, the issue on the relative controllability with respect to terminal state constraint for a class of linear time-varying impulsive systems has been addressed. Several types of criteria for relative controllability of such systems have been established respectively. Moreover, some corresponding necessary and sufficient conditions for controllability of linear time-invariant impulsive systems have also been obtained more compactly. Meanwhile, some equivalent relationships between different kinds of controllability are established and our criteria are compared with the existing results. The results obtained will be useful in the analysis and practical applications of impulsive systems. In our opinion, for following-up or future work on this topic, to be expected is to extend our approach to problems for relative controllability with respect to general terminal state constraints for semilinear and nonlinear impulsive systems without time-delays or with them. In the near future we are going to solve such issues.

References

[1] Z. Benzaid and M. Sznaier, Constrained controllability of linear impulse differential systems, IEEE Transactions on Automatic Control, 39, 5 (1994), 1064-1066.

[2] R. K. George, A. K. Nandakumaran and A. Arapostathis, A note on controllability of impulsive systems, Journal of Mathematical Analysis and Applications, 241 (2000), 276-283.

[3] Z. H. Guan, T. H. Qian and X. H. Yu, On controllability and observability for a class of impulsive systems, Systems & Control Letters, 47 (2002), 247-257.

[4] Z. H. Guan, T. H. Qian and X. H. Yu, Controllability and observability of linear time-varying impulsive systems, IEEE Transaction on Circuits and Systems, 49, 8 (2002), 1198-1208.

[5] S. C. Ji, G. Li and M. Wang, Controllability of impulsive differential systems with nonlocal conditions, Applied Mathematics and Computation, 217 (2011), 6981-6989.

[6] E. Joelianto and D. Williamson, Controllability of linear impulsive systems, Information, Decision and Control: Data and Information Fusion Symposium, Signal Processing and Communications Symposium and Decision and Control Symposium, Adelaide, SA, Australia, IEEE, 2002, 205-210.

[7] S. Leela, Farzana A. McRae and S. Sivasundaram, Controllability of impulsive differential equations, Journal of Mathematical Analysis and Applications, 177 (1993), 24-30.

[8] X. Z. Liu, Impulsive control and optimization, Applied Mathematics and Computation, 73 (1995), 77-98.

[9] B. M. Miller and E. Ya. Rubinovich, Impulsive control in continuous and discrete-continuous systems, Norwell, MA: Kluwer, 2003.

[10] S. Sivasundaram and Josaphat Uvah, Controllability of impulsive hybrid integro-differential systems, Nonlinear Analysis: Hybrid Systems, 2 (2008), 1003-1009.

[11] Vijayakumar S. Muni and Raju K. George, Controllability of semelinear impulsive control systems with multiple time delays in control, IMA Journal of Mathematical Control and Information, 00 (2018), 1-31.

[12] G. M. Xie and L. Wang, Necessary and sufficient conditions for controllability and observability of switched impulsive control systems, IEEE Transactions on Automatic Control, 49, 6 (2004), 960-966.

[13] G. M. Xie and L. Wang, Controllability and observability of a class of linear impulsive systems, Journal of Mathematical Analysis and Applications, 304 (2005), 336-355.

[14] D. G. Xu, Controllability and observability of a class of piecewise linear impulsive control systems, Advances in Computer, Communication, Control & Automation, LNEE 121 (2011), 321-328.

[15] T. Yang, Impulsive control theory, Springer, 2001.

[16] Z. L. You, M. Feˇckan, J. R. Wang and D. O'Regan, Relative controllability of impulsive multi-delay differential systems, Nonlinear Analysis: Modeling and Control, 27, 1 (2022), 70-90.

[17] S. W. Zhao and J. T. Sun, Controllability and obserbility for a class of time-varying impulsive systems, Nonlinear Analysis: Real World Applications, 10 (2009), 1370-1380.

[18] S. W. Zhao and J. T. Sun, Controllability and obserbility for impulsive systems in complex fields, Nonlinear Analysis: Real World Applications, 11 (2010), 1513-1521.

[19] S. W. Zhao, Z. H. Zhang, T. B. Wang and W. Q. Yu, Controllability for a class of time-varying controlled switching impulsive systems with time delays, Applied Mathematics and Computation, 228 (2014), 404-410.

[20] S. W. Zhao, Z. H. Zhang, T. B. Wang and W. Q. Yu, Controllability for a class of time-varying controlled switching impulsive systems with time delays, IEEE Transactions on Cercuits and Systems-I Fundamental Theory and Applications, 49, 8 (2002), 1198-1208.