

# A Weak Galerkin Finite Element Method for the Incompressible Viscous Magneto-hydrodynamic Equations

Su Sung Om<sup>1</sup>, Kwang Jin Kim<sup>1</sup>, Thae Gun Oh<sup>1</sup>, Nam Chol Yu<sup>2,\*</sup>

<sup>1</sup>Faculty of Mathematics, Kim Hyen Jik Education University, Pyongyang, Democratic People's Republic of Korea

<sup>2</sup>School of Science and Engineering, Kim Chaek University of Technology, Pyongyang, Democratic People's Republic of Korea

## Email address:

ync781213@star-co.net.kp (Nam Chol Yu)

\*Corresponding author

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**Abstract:** In this paper, we are studied the weak Galerkin finite element method for the incompressible viscous Magneto-hydrodynamic (MHD) equations. There have been the several numerical methods for the incompressible viscous Magneto-hydrodynamic equations: the continuous Galerkin finite element methods, the discontinuous Galerkin finite element methods, and the mixed Galerkin finite element methods et al. These numerical methods are converted to large-scale system of linear or nonlinear equations, which are solved by direct or iterative method. Besides accuracy, efficiency, and robustness of numerical methods, physical properties such as local conservation and flux continuity are very important in practical applications. And these numerical methods have following properties. (1) The continuous Galerkin finite element methods are known as lacking of “local conservation”. (2) The discontinuous Galerkin finite element methods are locally conservative but there is no continuity. Also the discontinuous Galerkin finite element methods increase in numbers of unknowns. (3) The mixed Galerkin finite element methods approximate the two variables that satisfy the inf-sup condition. Therefore the obtained saddle point problems are difficult to solve. A weak Galerkin finite element methods are locally conservative well, continuous across element interfaces, less unknowns than discontinuous Galerkin finite element methods, and definite discrete linear systems. A weak Galerkin finite element methods are based on new concept called discrete weak gradient, discrete weak divergence and discrete weak rotation, which are expected to play an important role in numerical methods for magneto-hydrodynamic equation. This article intends to provide a general framework for managing differential, divergence, rotation operators on generalized functions. With the proposed method, solving the magneto-hydrodynamic (MHD) equation is that the classical gradient, divergence, vortex operators are replaced by the discrete weak gradient, divergence, rotation and apply the Galerkin finite element method. It can be seen that the solution of the weak Galerkin finite element method is not only continuous function but also totally discontinuous function. For the proposed method, optimal order error estimates are established in Hilbert space.

**Keywords:** Weak Galerkin Finite Element Method, Incompressible Viscous Magneto-hydrodynamic Equations, Discrete Weak Gradient, Discrete Weak Divergence, Discrete Weak Vortex, Navier-Stokes Equations

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## 1. Introduction

Magneto-hydrodynamic (MHD) equations which have been widely used in industry and engineering, such as liquid metal cooling of nuclear reactors, process metallurgy, simulate aluminum electrolysis cells and so on are composed

of Navier-Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. These equations are concerned with the viscous, incompressible, electrically conducting fluid and an external magnetic field. In this paper, we consider the magneto-hydrodynamic equations.

$$\frac{\partial \mathbf{u}}{\partial t} - \gamma \cdot \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = a \cdot \text{rot} \mathbf{B} \times \mathbf{B}, \quad \mathbf{x} \in \Omega, \quad t \in (0, T) \quad (1)$$

$$\frac{\partial \mathbf{B}}{\partial t} + \text{rot} \mathbf{E} = 0, \quad \mathbf{j} = \text{rot} \mathbf{B} = \frac{1}{\gamma_\mu} (\mathbf{E} + \mathbf{u} \times \mathbf{B} + \mathbf{J}_c) \quad (2)$$

$$\text{div} \mathbf{u} = 0, \quad \text{div} \mathbf{B} = 0 \quad (3)$$

with the following initial condition and homogeneous Dirichlet boundary conditions.

$$\mathbf{u}|_{t=0} = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{B}|_{t=0} = \mathbf{B}_0(\mathbf{x}) \quad \mathbf{x} \in \Omega \quad (4)$$

$$\mathbf{u} = 0, \quad \mathbf{B} \cdot \mathbf{n} = 0, \quad \mathbf{E} \times \mathbf{n} = 0, \quad \mathbf{J}_c \times \mathbf{n} = 0 \quad (\mathbf{x}, t) \in \partial\Omega \times (0, T) \quad (5)$$

where  $\Omega \subset \mathbb{R}^n$  ( $n = 2, 3$ ) is a bounded-closed, convex domain with Lipschitz-continuous boundary  $\partial\Omega$ ,  $\mathbf{u}$ ,  $\mathbf{B}$ ,  $\mathbf{E}$ ,  $\mathbf{j}$  are velocity, magnetic field, intensity of electric field, electric current density,  $\mathbf{J}_c$  is minor control possible electromotive force,  $p$  is hydrodynamic pressure,

$\gamma = \frac{1}{R_e}$ ,  $\gamma_\mu = \frac{1}{R_\mu}$ ,  $a = \frac{M^2}{R_e R_\mu}$  and the three parameter

$R_e$ ,  $R_\mu$ ,  $M$  represent hydrodynamic Reynolds number, magnetic Reynolds number and Hartmann number. Because MHD equations have practical significance in many fields of science, technology and production, many researchers gave some research results for magneto-hydrodynamic. [1, 2, 7, 9].

Some research results are given as follows:

In [3–5], the mixed and stabilized FE methods were used to solve MHD equations.

Additionally, Y. He [8] studied an unconditional convergence of the Euler semi-implicit scheme and G. Yuksel and R. Ingram [6] investigated a full discretization of Crank–Nicolson scheme for the non-stationary MHD equations with small magnetic Reynolds numbers.

In [10], it was studied the design and analysis of some structure preserving finite element schemes for the magneto-hydrodynamics (MHD) system.

X. Feng et al have applied some Uzawa-type iterative algorithms to the steady magneto-hydrodynamic (MHD) equations discretized by mixed finite element method [11].

In [12], it has focused on a fractional-step finite element method for the magneto-hydrodynamics problems in three-dimensional bounded domains.

In [13], the convergence analyses of standard Galerkin finite element method and a new highly efficient two-step algorithm for the stationary incompressible magneto-hydrodynamic equations was studied.

In [14], it was devoted to extension of boundary element method (BEM) for solving coupled equations in velocity and induced magnetic field for time dependent magneto-hydrodynamic (MHD) flows through a rectangular pipe.

In [15], Y. Rong and Y. Hou have studied a partitioned scheme based on Gauge-Uzawa finite element method for the 2D time-dependent incompressible magneto-hydrodynamics (MHD) equations.

In this paper, we are going to propose a formulation for the weak Galerkin finite element method for the magneto-hydrodynamic (MHD) equations (1)–(5).

The weak Galerkin (WG) method was recently introduced in [16] for second-order elliptic problems based on local RT elements.

It is an extension of the standard Galerkin finite element method where classical operators (e.g., gradient, divergence, and curl) are substituted by weakly defined operators.

Then, in [17], the weak Galerkin method was extended to allow arbitrary shapes of finite elements in a partition by adding parameter free stabilizer, which enforces a certain weak continuity and provides a convenient flexibility in mesh generation.

Through rigorous analysis, the optimal order of priori error estimates has been established for various weak Galerkin discretization schemes for second order elliptic equation in [16–18].

And the possibility of an optimal combination of polynomial spaces that minimizes the number of unknowns has been explored in several numerical experiments in [19].

On the base of the weak Galerkin mixed finite element methods, the weak Galerkin method for the Stokes equations was stated in [20].

In [28], it was considered the lowest-order weak Galerkin method for linear elasticity based on the displacement formulation.

Moreover, because the weak Galerkin method inherits the advantages and abandons the weaknesses of a discontinuous Galerkin or discontinuous Petrov–Galerkin method, it has been developed to solve many equations.

Thus, we refer to several papers for applications of the WG method to some other partial differential equations, such as, elliptic interface equations [31], Oseen equations [21], Helmholtz equations [22, 23], Darcy–Stokes equations [24, 29, 30], convection–diffusion–reaction equations [32], Sobolev equation [33] and parabolic equations [25–27] etc.

It is well known that the magneto-hydrodynamic equations involve a trilinear term, which changes the essence of all the problems considered so far (linear or nonlinear problems).

The goal of this article are to construct and analyze a stable weak Galerkin finite element method for the magneto-hydrodynamic equations (1)–(5) by using the definition of a weak trilinear term.

This method allows the use of finite element partitions with arbitrary shapes of polygons or polyhedra with shape

regularity and parameter free.

An outline of this paper is as follows. In the next section, we introduce some notations for the magneto-hydrodynamic equations (1)–(5) and Sobolev spaces. In Section 3, the fundamental definitions and weak Galerkin finite element scheme for the magneto-hydrodynamic equations are developed. Then, in Section 4, we estimate the error of weak Galerkin finite element approximation solution for the incompressible viscous Magneto-hydrodynamic equations.

In Section 5, we give numerical experiment to verify the studied theoretical analysis. Finally, conclusions are drawn in Section 6.

## 2. Preliminary Results and Notations

From the systems of equations (1), (2).

$$\frac{\partial \mathbf{u}}{\partial t} - \gamma \cdot \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p - \mathbf{a} \cdot \text{rot} \mathbf{B} \times \mathbf{B} = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} + \gamma_\mu \cdot \text{rot} \cdot \text{rot} \mathbf{B} - \text{rot}(\mathbf{u} \times \mathbf{B}) = \text{rot} \mathbf{J}_c$$

$$\text{div} \mathbf{u} = 0, \quad \text{div} \mathbf{B} = 0$$

Therefore, we consider the following non-stationary incompressible magneto-hydrodynamic equations.

$$\frac{\partial \mathbf{u}}{\partial t} - \gamma \cdot \Delta \mathbf{u} + \text{rot} \mathbf{u} \times \mathbf{u} - \mathbf{a} \cdot \text{rot} \mathbf{B} \times \mathbf{B} + \nabla(p + \frac{1}{2} |\mathbf{u}|^2) = 0 \quad (x, t) \in \Omega \times (0, T), \quad (6)$$

$$\frac{\partial \mathbf{B}}{\partial t} - \gamma_\mu \text{rot} \cdot \text{rot} \mathbf{B} - \text{rot}(\mathbf{u} \times \mathbf{B}) = \text{rot} \mathbf{J}_c \quad (x, t) \in \Omega \times (0, T), \quad (7)$$

$$\text{div} \mathbf{u} = 0, \quad \text{div} \mathbf{B} = 0 \quad (x, t) \in \Omega \times (0, T), \quad (8)$$

with the initial-boundary conditions

$$\mathbf{u}|_{t=0} = \mathbf{u}_0(x), \quad \mathbf{B}|_{t=0} = \mathbf{B}_0(x) \quad x \in \Omega \quad (9)$$

$$\mathbf{u} = 0, \quad \mathbf{B} \cdot \mathbf{n} = 0, \quad \mathbf{J}_c \times \mathbf{n} = 0 \quad (x, t) \in \partial\Omega \times (0, T) \quad (10)$$

We introduce the following notations of some norms and spaces:

$$L^2(\Omega) = [L^2(\Omega)]^d, \quad W_2^1 = [W_2^1(\Omega)]^d, \quad (d = 2, 3)$$

$$H(\text{div}, \Omega) = \{v \mid v \in L^2(\Omega), \nabla \cdot v \in L^2(\Omega)\}$$

$$H(\text{rot}, \Omega) = \{v \mid v \in L^2(\Omega), \nabla \times v \in L^2(\Omega)\}$$

$$H_1 = H_2 = \{w \in L^2(\Omega) \mid \text{div} w = 0, \quad w \cdot \mathbf{n}|_{\partial\Omega} = 0\}$$

$$V_1 = \{u \in W_2^1(\Omega) \mid \text{div} u = 0, \quad u|_{\partial\Omega} = 0\},$$

$$V_2 = \{B \in W_2^1(\Omega) \mid \text{div} B = 0, \quad B \cdot \mathbf{n}|_{\partial\Omega} = 0\}$$

$$V_3 = H_0(\text{rot}, \Omega) = \{u \in L^2(\Omega) \mid \text{rot} u \in L^2(\Omega), \quad u \times \mathbf{n}|_{\partial\Omega} = 0\}$$

$$(u, v) = \int_{\Omega} u \cdot v \, dx, \quad \|u\| = \sqrt{(u, u)}, \quad \forall u, v \in H_1 \text{ or } H_2$$

$$[u, v] = (\text{rot} u, \text{rot} v) = \int_{\Omega} \text{rot} u \cdot \text{rot} v \, dx, \quad |u| = \sqrt{[u, u]}, \quad \forall u, v \in V_1 \text{ or } V_2$$

$$H = H_1 \times H_2, \quad V = V_1 \times V_2, \quad V \subset H = H' \subset V'$$

where  $(\text{rot} u, \text{rot} v) = (\nabla u, \nabla v) \quad \forall u, v \in V_1$  and  $H', V'$  are dual spaces of  $H, V$ .

We define the operators  $A_1: V_1 \rightarrow V_1$ ,  $A_2: V_2 \rightarrow V_2$ ,  $A: V \rightarrow V'$ ,  $G: V \times V \rightarrow V'$ ,  $F: L^2(\Omega) \rightarrow V'$  following as:

$$(Ay, z) = \gamma(A_1 u, v) + \gamma_\mu(A_2 B, w) = a_1(u, v) + a_2(B, w)$$

$$a_1(u, v) = \gamma(\nabla u, \nabla v)$$

$$a_2(B, w) = \gamma_\mu(\text{rot} B, \text{rot} w)$$

$$(G(y_1, y_2), z) = (\text{rot} u_1 \times u_2 - \text{rot} B_2 \times B_1, v) - (u_2 \times B_1, \text{rot} w)$$

$$(F(J_c), z) = (J_c, \text{rot} w),$$

where  $y = \{u, B\}$ ,  $y_1 = \{u_1, B_1\}$ ,  $y_2 = \{u_2, B_2\}$ ,  $z = \{v, w\} \in V$ ,  $J_c \in L^2(\Omega)$

For the operator  $A: V \rightarrow V'$ ,

$$(Ay, y) \geq \alpha |y|^2, \quad \alpha = \min\{\gamma, \gamma_\mu\}$$

$$(Ay, z) = (y, Az), \quad \forall y, z \in V$$

For the operator  $G: V \times V \rightarrow V'$ , representing  $G(y, y) = G[y]$ ,

$$(G(y, z), z) = 0, \quad (G[y], z) = (\text{rot} u \times u, v) - (\text{rot} B \times B, v) - (u \times B, \text{rot} w).$$

By the Hölder's inequality, we can know that norm satisfies  $|u|_{L^4(\Omega)} \leq K \|u\|^{\frac{1}{4}} \cdot |u|^{\frac{3}{4}}$ .

For operator  $G: V \times V \rightarrow V'$ , the trilinear form  $(G(y_1, y_2), y_3)$  satisfies.

$$(G(y_1, y_2), y_3) \leq C_1 |y_1|^{\frac{3}{4}} \cdot \|y_1\|^{\frac{1}{4}} \cdot |y_2|^{\frac{3}{4}} \cdot \|y_3\|^{\frac{1}{4}}, \quad y_1, y_2, y_3 \in V, \quad (11)$$

$$(G(y_1, y_2), y_3) \leq C_1 |y_1| \cdot |y_2|^{\frac{1}{2}} \cdot \|Ay_2\|^{\frac{1}{2}} \cdot \|y_3\|, \quad y_1 \in V, \quad y_2 \in D(A), \quad y_3 \in H, \quad (12)$$

where  $C_1 > 0$  is a constant independent of  $\Omega, R_e, R_\mu$ . [34].

The weak formulation of problem (6) - (10) can be written a variation forms as follows.

$$(\dot{y}, z) + (Ay, z) + (G[y], z) = (F(J_c), z) \quad \forall z \in V, \quad \forall t \in (0, T), \quad (13)$$

$$y(0) = y_0 \quad \forall z \in V, \quad (14)$$

where  $y = \{u, B\}$  is called a weak solution if  $y \in L^2(0, T; V)$

and  $\dot{y} \in L^2(0, T; V')$  are the solutions of equations (13) – (14).

Next, we will introduce the weak gradient operator, weak divergence operator and newly weak rotation operator defined on a space of generalized functions.

To explain weak gradient, weak divergence and weak rotation, let  $K$  be any polygonal domain with interior  $K^0$  and boundary  $\partial K$ .

A weak function on the region  $K$  refers to a function  $v = \{v_0, v_b\}$  such that  $v_0 \in L^2(K)$  and

$$v_b \in H^{\frac{1}{2}}(\partial K) = \left[ H^{\frac{1}{2}}(\partial K) \right]^d, \quad \text{where the first component } v_0$$

can be understood as the value of  $v$  in the interior of  $K$  and the second component  $v_b$  is the value of  $v$  on the boundary of  $K$ .

Denote by  $W(K)$  the space of weak functions associated with  $K$ ; i.e.,

$$W(K) = \left\{ v = \{v_0, v_b\} \mid v_0 \in L^2(K), v_b \in H^{\frac{1}{2}}(\partial K) \right\} \quad (15)$$

The dual of  $L^2(K)$  can be identified with itself by using the standard  $L^2$  inner product as the action of linear functional.

Definition 1. ([16]) For any  $v \in W(K)$ , the weak gradient of  $v$  is defined as a linear function  $\nabla_w v$  in the dual space of  $H(\text{div}, K)$  whose action on each  $q \in H(\text{div}, K)$  is given by

$$(\nabla_w v, q)_K = -(v_0, \nabla \cdot q)_K + \langle v_b, q \cdot n \rangle_{\partial K},$$

where  $n$  is the outward normal direction to  $\partial K$ .

The discrete weak gradient operator denoted by  $\nabla_{w,r,K}$  is defined as the unique polynomial  $\nabla_{w,r,K} v \in [P_r(K)]^{d \times d}$  satisfying the following equation:

$$(\nabla_{w,r,K} v, w)_K = -(v_0, \text{div} w)_K + \langle v_b, w \cdot n \rangle_{\partial K}, \quad \forall w \in [P_r(K)]^{d \times d} \quad (16)$$

Definition 2.([16]). For any  $v \in W(K)$ , the weak divergence of  $v$  is defined as a linear function  $\text{div}_w v$  in the dual space of  $W_2^1(K)$  whose action on each  $q \in W_2^1(K)$  is given by

$$(\text{div}_w v, q)_K = -(v_0, \nabla q)_K + \langle v_b \cdot n, q \rangle_{\partial K},$$

where  $n$  is the outward normal direction to  $\partial K$ .

The discrete weak divergence operator denoted by  $\text{div}_{w,r,K}$  is defined as the unique polynomial  $\text{div}_{w,r,K} v \in P_r(K)$  satisfying the following equation:

$$(\text{div}_{w,r,K} v, q)_K = -(v_0, \nabla q)_K + \langle v_b \cdot n, q \rangle_{\partial K}, \quad \forall q \in P_r(K) \quad (17)$$

Then we introduce the definition of a weak rotation operator.

Definition 3. For any  $v \in W(K)$ , the weak rotation of  $v$  is defined as a linear function  $\text{rot}_w v$  in the dual space of  $H(\text{rot}, K)$  whose action on each  $q \in H(\text{rot}, K)$  is given by

$$(\text{rot}_w v, q)_K = (v_0, \text{rot} q)_K - \langle v_b \times n, q \rangle_{\partial K},$$

where  $n$  is the outward normal direction to  $\partial K$ .

The discrete weak rotation operator denoted by  $\text{rot}_{w,r,K}$  is defined as the unique polynomial  $\text{rot}_{w,r,K} v \in [P_r(K)]^d$  satisfying the following equation:

$$(\text{rot}_{w,r,K} v, q)_K = (v_0, \text{rot} q)_K - \langle v_b \times n, q \rangle_{\partial K}, \quad \forall q \in [P_r(K)]^d \quad (18)$$

### 3. Weak Galerkin Finite Element Method (WGFEM)

In this section, we design a continuous time and discontinuous time WGFEM for the problem (13)–(14).

Let  $K_h$  be a regular, quasi-uniform partition of the domain

$\Omega$  and  $K \in K_h$  be any polygonal domain with interior  $K^0$  and boundary  $\partial K$ , where the mesh size  $h = \max h_K$ ,  $h_K$  is the diameter of element  $K$ .

Then, we can introduce the discrete weak Galerkin finite element spaces on a given mesh: for the velocity variable,

$$V_{1h} = \left\{ u_h = \{u_0^h, u_b^h\} \mid u_0^h|_K \in [P_i(K)]^d, u_b^h|_{\partial K} \in [P_{i-1}(K)]^d \text{ for all } K \in K_h \right\}$$

and denote  $V_{1h}^0 = \left\{ u_h = \{u_0^h, u_b^h\} \mid u_h \in V_{1h}, u_b^h|_{\partial K \cap \partial \Omega} = 0 \right\}$ ; for the magnetic field variable,

$$V_{2h} = \left\{ B_h = \{B_0^h, B_b^h\} \mid B_0^h|_K \in [P_i(K)]^d, B_b^h|_{\partial K} \in [P_{i-1}(K)]^d \text{ for all } K \in K_h \right\}$$

and denote  $V_{2h}^0 = \left\{ B_h = \{B_0^h, B_b^h\} \mid B_h \in V_{2h}, B_b^h \cdot n|_{\partial K \cap \partial \Omega} = 0 \right\}$ .

Moreover, we denote  $y_h = \{y_0^h, y_b^h\}$  by  $y_0^h = \{u_0^h, B_0^h\}$ ,  $y_b^h = \{u_b^h, B_b^h\}$ .

In the further, we shall drop the subscript  $r$  and  $K$  to simplify the notations for the discrete weak gradient, divergence and rotation operators.

To investigate the approximation properties of the discrete weak Galerkin finite element spaces  $V_{1h}$  and  $V_{2h}$ , we use

$$(\text{div} v, v_0)_K = (\text{div} \Pi_h v, v_0)_K \quad \forall v_0 \in P_i(K) \quad (19)$$

For  $y = \{u, B\} \in V = V_1 \times V_2$ , we set  $Q_h y = \{Q_h u, Q_h B\}$ ,  $R_h y = \{R_h u, R_h B\}$ ,  $R_h y = \{R_h u, R_h B\}$  and  $\Pi_h y = \{\Pi_h u, \Pi_h B\}$ .

three projection operators:  $Q_h v = \{Q_0 v, Q_b v\}$  is  $L^2$  projection operator from  $W_2^1$  onto  $V_{1h}$  or  $V_{2h}$ ,  $R_h v$  is  $L^2$  projection operator onto  $[P_i(K)]^{d \times d}$ ,  $R_h v$  is  $L^2$  projection operator onto  $P_i(K)$  and  $\Pi_h v$  is  $L^2$  projection operator from  $H(\text{div}, \Omega)$  onto  $H(\text{div}, \Omega)$ ,  $\Pi_h v \in [P_i(K)]^d$  satisfies.

The  $L^2$  projection operators  $Q_h$ ,  $R_h$  and  $R_h$  satisfy the following properties.

Lemma 3.1. For any  $y = \{u, B\} \in V = V_1 \times V_2$ , The  $L^2$  projection operators  $Q_h$ ,  $R_h$  and  $R_h$  satisfy

$$\nabla_w Q_h y = R_h(\nabla y), \quad \text{div}_w Q_h y = R_h(\text{div} y), \quad \text{rot}_w Q_h y = R_h(\text{rot} y) \quad (20)$$

$$\|y - Q_h^0 y\|_{0,K} \leq Ch_K^s \|y\|_{s,K}, \quad 0 \leq s \leq i, \quad K \in K_h, \quad (21)$$

$$\|\nabla_w Q_h y - \nabla y\|_{0,K} \leq Ch_K^s \|y\|_{s+1,K}, \quad 0 \leq s \leq i+1, \quad K \in K_h \quad (22)$$

Proof. From Definition 1,  $L^2$  projection operator  $Q_h y$  and Green's formula, we have

$$\int_K \nabla_w Q_h y \cdot q dx = - \int_K (Q_0 y) \nabla \cdot q dx + \int_{\partial K} (Q_b y) q \cdot n ds, \quad \forall q \in [P_i(K)]^{d \times d} \quad (23)$$

Since  $Q_0$  and  $Q_b$  are  $L^2$ -projection operator, then the right-hand side of (23) is given by

$$- \int_K (Q_0 y) \nabla \cdot q dx + \int_{\partial K} (Q_b y) q \cdot n ds = - \int_K y \nabla \cdot q dx + \int_{\partial K} y q \cdot n ds = \int_K (\nabla y) \cdot q dx = \int_K R_h(\nabla y) \cdot q dx$$

This shows that  $\nabla_w Q_h y = R_h(\nabla y)$  holds.

Similarly, from Definition 2, we can derive  $\text{div}_w Q_h y = R_h(\text{div} y)$  and  $\text{rot}_w Q_h y = R_h(\text{rot} y)$ .

Also, it implies approximation property (21) and estimate (22).

Lemma 3.2. For any  $q \in H(\text{div}, \Omega) \times H(\text{div}, \Omega)$ ,

$$\sum_{K \in K_h} (-\nabla \cdot q, y_0)_K = \sum_{K \in K_h} (\Pi_h q, \nabla_w y)_K, \quad \forall y = (y_0, y_b) \in V_{1h}^0 \times V_{2h}^0 \quad (24)$$

Proof. From  $L^2$  projection  $\Pi_h$  and Definition 1,

$$\sum_{K \in K_h} (-\nabla \cdot q, y_0)_K = \sum_{K \in K_h} (-\nabla \cdot \Pi_h q, y_0)_K = \sum_{K \in K_h} (\Pi_h q, \nabla_w y)_T - \sum_{K \in K_h} (y_b, \Pi_h q \cdot n)_{\partial K} = \sum_{K \in K_h} (\Pi_h q, \nabla_w y)_K$$

The weak Galerkin finite element method is to replace the classical gradient and rotation operators by the weak gradient operator  $\nabla_w$  and weak rotation operator  $\text{rot}_w$  and to use the discrete weak finite element space  $V_{1h}^0 \times V_{2h}^0$ .

First, we introduce the semi-discrete WG finite element method.

The semi-discrete WG finite element method of (13)–(14) is to find  $y_h(x, t) \in V_{1h}^0 \times V_{2h}^0$ , satisfying  $u_b^h = 0$  and  $B_b^h \cdot n = 0$  on  $\partial\Omega$ ,  $t \in [0, T]$  for  $Q_b y = y_b^h = \{u_b^h, B_b^h\}$ ,  $y_h(x, 0) = Q_h y_0$  in  $\Omega$  and the following equation.

$$(\dot{y}_h, z_0) + A(y_h, z) + (G[y_h], z) = (F(J_c), z_0), \quad \forall z \in V_{1h}^0 \times V_{2h}^0, \quad t \in [0, T] \quad (25)$$

where  $(Ay, z)$  and  $(G[y], z)$  are respectively the weak bilinear and trilinear form defined by

$$\begin{aligned} A(y, z) &= a_1(u, v) + a_2(B, w) \\ a_1(u, v) &= \gamma(\nabla_w u, \nabla_w v) = \gamma(\text{rot}_w u, \text{rot}_w v) \\ a_2(B, w) &= \gamma_\mu(\text{rot}_w B, \text{rot}_w w) \end{aligned}$$

$$(G_w[y], z) = (\text{rot}_w u \times u - \text{rot}_w B \times B, v) - (u \times B, \text{rot}_w w)$$

and  $(F(J_c), z_0)$  is defined by  $(F(J_c), z_0) = (J_c, \text{rot}_w w_0)$ .

For the equation (25), we can get the following splitting equation, respectively:

$$\begin{cases} \left( \frac{\partial \mathbf{u}_h}{\partial t}, \mathbf{v}_0 \right) + \gamma \cdot (\nabla_w \mathbf{u}_h, \nabla_w \mathbf{v}) + (\text{rot}_w \mathbf{u}_h \times \mathbf{u}_h, \mathbf{v}) - (\text{rot}_w \mathbf{B}_h \times \mathbf{B}_h, \mathbf{v}) = 0 \\ \left( \frac{\partial \mathbf{B}_h}{\partial t}, \mathbf{w}_0 \right) + \gamma_\mu (\text{rot}_w \mathbf{B}_h, \text{rot}_w \mathbf{w}) - (\mathbf{u}_h \times \mathbf{B}_h, \text{rot}_w \mathbf{w}) = (\mathbf{J}_c, \text{rot}_w \mathbf{w}_0), \end{cases} \quad (26)$$

where  $(\nabla_w \mathbf{u}, \nabla_w \mathbf{v}) = \sum_{K \in K_h} (\nabla_w \mathbf{u}, \nabla_w \mathbf{v})_K$ ,  $(\text{rot}_w \mathbf{B}, \text{rot}_w \mathbf{w}) = \sum_{K \in K_h} (\text{rot}_w \mathbf{B}, \text{rot}_w \mathbf{w})_K$ ,  $(\text{rot}_w \mathbf{u}, \mathbf{v}) = \sum_{K \in K_h} (\text{rot}_w \mathbf{u}, \mathbf{v})_K$ .

Next, we turn our attention to the full discrete WG finite element method.

Let  $\tau$  be the time step size and  $t_n = n\tau$ , where  $n$  is a nonnegative integer.

We denote by  $y_h(t_n) = y_h^n$  the approximation of  $y(t_n)$ .

Writing the full discrete WG finite element scheme for equation (13)–(14),

$$(\partial_t y_h^n, z_0) + A(y_h^n, z) + (G_w[y_h^n], z) = (F(\mathbf{J}_c), z_0), \quad \forall z \in \mathbf{V}_{1h}^0 \times \mathbf{V}_{2h}^0, y_h^0 = Q_h y_0 \quad (27)$$

where  $\partial_t y_h^k = \frac{y_h^{k+1} - y_h^k}{\tau}$  or  $\partial_t y_h^k = \frac{y_h^k - y_h^{k-1}}{\tau}$ .

For the equation (27), we can get the following splitting equation, respectively:

$$\begin{cases} \left( \frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}}{\tau}, \mathbf{v}_0 \right) + \gamma \cdot (\nabla_w \mathbf{u}_h^n, \nabla_w \mathbf{v}) + (\text{rot}_w \mathbf{u}_h^n \times \mathbf{u}_h^n, \mathbf{v}) - (\text{rot}_w \mathbf{B}_h^n \times \mathbf{B}_h^n, \mathbf{v}) = 0 \\ \left( \frac{\mathbf{B}_h^n - \mathbf{B}_h^{n-1}}{\tau}, \mathbf{w}_0 \right) + \gamma_\mu (\text{rot}_w \mathbf{B}_h^n, \text{rot}_w \mathbf{w}) - (\mathbf{u}_h^n \times \mathbf{B}_h^n, \text{rot}_w \mathbf{w}) = (\mathbf{J}_c, \text{rot}_w \mathbf{w}_0) \\ \mathbf{u}_h^0 = Q_h \mathbf{u}_0, \quad \mathbf{B}_h^0 = Q_h \mathbf{B}_0 \end{cases} \quad (28)$$

or

$$\begin{cases} \left( \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\tau}, \mathbf{v}_0 \right) + \gamma \cdot (\nabla_w \mathbf{u}_h^n, \nabla_w \mathbf{v}) + (\text{rot}_w \mathbf{u}_h^n \times \mathbf{u}_h^n, \mathbf{v}) - (\text{rot}_w \mathbf{B}_h^n \times \mathbf{B}_h^n, \mathbf{v}) = 0 \\ \left( \frac{\mathbf{B}_h^{n+1} - \mathbf{B}_h^n}{\tau}, \mathbf{w}_0 \right) + \gamma_\mu (\text{rot}_w \mathbf{B}_h^n, \text{rot}_w \mathbf{w}) - (\mathbf{u}_h^n \times \mathbf{B}_h^n, \text{rot}_w \mathbf{w}) = (\mathbf{J}_c, \text{rot}_w \mathbf{w}_0) \\ \mathbf{u}_h^0 = Q_h \mathbf{u}_0, \quad \mathbf{B}_h^0 = Q_h \mathbf{B}_0 \end{cases} \quad (29)$$

This approximate solutions  $\mathbf{u}_h^k$  and  $\mathbf{B}_h^k$  are called the approximate solutions of the full discrete WG finite element method for the equations (13)–(14).

Remark 1. In this article, we haven't indicated for the hydrodynamic pressure  $p = p(x, t)$ , because we can find easily the hydrodynamic pressure  $p = p(x, t)$  from the equation (6) if we have found out the approximate solutions  $\mathbf{u}_h^k$  and  $\mathbf{B}_h^k$  in the equation (26) or (28).

## 4. Error Analysis of Approximate Solution

In this section, we derive some error estimates for semi-discrete and full discrete WG finite element methods.

By means of projections  $Q_h$  and  $R_h$ , we can derive the following approximation property.

Lemma 4.1. For  $\mathbf{u}, \mathbf{B} \in \mathbf{H}^{1+s}(\Omega)$  with  $s > 0$ , we have respectively.

$$\|\Pi_h(\nabla \mathbf{u}) - \nabla_w(Q_h \mathbf{u})\| \leq ch^s \|\mathbf{u}\|_{1+s}, \quad \|\Pi_h(\text{rot} \mathbf{B}) - \text{rot}_w(Q_h \mathbf{B})\| \leq ch^s \|\mathbf{B}\|_{1+s} \quad (30)$$

$$\|\nabla \mathbf{u} - \nabla_w(Q_h \mathbf{u})\| \leq ch^s \|\mathbf{u}\|_{1+s}, \quad \|\text{rot} \mathbf{B} - \text{rot}_w(Q_h \mathbf{B})\| \leq ch^s \|\mathbf{B}\|_{1+s} \quad (31)$$

Proof. Since from (20) we have  $\nabla_w Q_h \mathbf{u} = R_h(\nabla \mathbf{u})$  and  $\nabla_w Q_h \mathbf{B} = R_h(\nabla \mathbf{B})$ , then

$$\|\Pi_h(\nabla u) - \nabla_w(Q_h u)\| = \|\Pi_h(\nabla u) - R_h(\nabla u)\|$$

Using the triangle inequality and the definition of  $\Pi_h$ , we have

$$\|\Pi_h(\nabla u) - R_h(\nabla u)\| \leq \|\Pi_h(\nabla u) - \nabla u\| + \|\nabla u - R_h(\nabla u)\| \leq c_1 h^{1+s} \|\nabla u\|_{1+s} + c_2 h^{1+s} \|\nabla u\|_{1+s} \leq c h^s \|u\|_{1+s}$$

Similarly, we can derive  $\|\Pi_h(\text{rot} B) - \text{rot}_w(Q_h B)\| \leq c h^s \|B\|_{1+s}$ .

The estimate (4.2) can be derived in a similar way. This completes a proof of the lemma.

Next, we shall prove the following estimate for the error of the semi-discrete solution.

Next, we introduce following lemma. [26].

Lemma 4.2. Assume that  $y = \{u, B\} \in V_{1h}^0 \times V_{2h}^0$  ( $y_0 = \{u_0, B_0\}$ ,  $y_b = \{u_b, B_b\}$ ), then there exist a constant  $C$  such that

$$\|y\| \leq C \|\text{rot}_w y\| \quad (32)$$

where  $\nabla_w y \in V_{1h} \times V_{2h}$ .

Theorem 4.3. Let  $y \in H^{1+s}(\Omega)$  and  $y_h$  be the solutions of equations (13)–(14) and (25), respectively. Then, there exists a constant  $\tilde{c}$  such that

$$\|y_h(t) - Q_h y(t)\|^2 \leq \tilde{c} h^{2s} \int_0^t \|y\|_{1+s}^2 dt \quad (33)$$

where  $y_h(t) = \{u_h(t), B_h(t)\}$ ,  $Q_h y(t) = \{Q_h u, Q_h B\}$ .

Proof. Let  $z \in \{z_0, z_b\} \in V_{1h}^0 \times V_{2h}^0$  be the testing function.

For  $y \in H^{1+s}(\Omega)$ , we know that  $\text{rot}_w(Q_h y) = R_h(\text{rot} y)$ ,

$(Q_h \dot{y}, z_0) = (\dot{y}, z_0)$  and  $(Q_h y, z_0) = (y, z_0)$ .

For  $y \in H^{1+s}(\Omega)$ , we obtain

$$\begin{aligned} (F(J_c), z_0) &= (\dot{y}, z_0) + \sum_{K \in K_h} A(y, z)_K + (G_w(y, y), z) = (\dot{y}, z_0) + (\Pi_h(\text{rot} y), \text{rot}_w z) + (G_w(y, y), z) \\ &= (Q_h \dot{y}, z_0) + (\text{rot}_w(Q_h y), \text{rot}_w z) + (\Pi_h(\text{rot} y) - R_h(\text{rot} y), \nabla_w z) + (G_w(Q_h y, Q_h y), z). \end{aligned}$$

Also, as the solution  $y_h$  of equation (25), we have

$$(F(J_c), z_0) = (\dot{y}_h, z_0) + (\text{rot}_w y_h, \text{rot}_w z) + (G_w(y_h, y_h), z)$$

Combining the above two equations, we get

$$(\dot{y}_h - Q_h \dot{y}, z_0) + (\text{rot}_w(y_h - Q_h y), \text{rot}_w z) = (\Pi_h(\text{rot} y) - R_h(\text{rot} y), \nabla_w z) + (G_w(Q_h y, Q_h y), z) - (G_w(y_h, y_h), z) \quad (34)$$

Now, Denote by  $e_h := y_h - Q_h y$  the difference between the weak Galekin approximation and the  $L^2$  projection of the exact solution  $y$ .

Substituting  $e_h$  for  $z$  in (34) and using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} (\dot{e}_h, e_{h0}) + (\text{rot}_w e_h, \text{rot}_w e_h) &= (\Pi_h(\text{rot} y) - R_h(\text{rot} y), \text{rot}_w e_h) + (G_w(Q_h y, Q_h y), e_h) \\ &\quad - (G_w(y_h, Q_h y), e_h) + (G_w(y_h, Q_h y), e_h) - (G_w(y_h, y_h), e_h) \\ &= (\Pi_h(\text{rot} \nabla y) - R_h(\text{rot} y), \text{rot}_w e_h) - (G_w(e_h, Q_h y), e_h) \\ &\leq \|\Pi_h(\text{rot} y) - R_h(\text{rot} y)\| \cdot \|\text{rot}_w e_h\| + c |e_h|^{\frac{3}{4}} \cdot \|e_h\|^{\frac{1}{4}} \cdot |Q_h y| \cdot |e_h|^{\frac{3}{4}} \cdot \|e_h\|^{\frac{1}{4}} \\ &\leq \|\Pi_h(\text{rot} y) - R_h(\text{rot} y)\| \cdot \|\text{rot}_w e_h\| + c |e_h|^{\frac{3}{2}} \cdot \|e_h\|^{\frac{1}{2}} \cdot |Q_h y| \\ &\leq \|\Pi_h(\text{rot} y) - R_h(\text{rot} y)\| \cdot \|\text{rot}_w e_h\| + c \|\text{rot}_w e_h\|^{\frac{3}{2}} \cdot \|e_h\|^{\frac{1}{2}} \cdot \|y\| \\ &\leq \|\Pi_h(\text{rot} y) - R_h(\text{rot} y)\|^2 + \frac{1}{4} \|\text{rot}_w e_h\|^2 + \frac{3}{4} \|\text{rot}_w e_h\|^2 + \frac{1}{4} c_1 \|e\|^2 \\ &= \|\Pi_h(\text{rot} y) - R_h(\text{rot} y)\|^2 + \|\text{rot}_w e_h\|^2 + c_2 \|e_h\|^2 \end{aligned}$$

Therefore, we have



$$\frac{1}{2} \frac{d}{dt} \|\mathbf{e}_h\|^2 + \|\text{rot}_w \mathbf{e}_h\|^2 \leq \|\Pi_h(\text{rot}y) - \mathbf{R}_h(\text{rot}y)\|^2 + \|\text{rot}_w \mathbf{e}_h\|^2 + c_2 \|\mathbf{e}_h\|^2.$$

We obtain by Lemma 4.1.

That is,

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{e}_h\|^2 \leq c_2 \|\mathbf{e}_h\|^2 + c'h^{2s} \|y\|_{s+1}^2. \quad (35)$$

$$\|\mathbf{e}_h\|^2 \leq \tilde{c} \cdot h^{2s} \int_0^t \|y\|_{s+1}^2 dt$$

Thus, integrating with respect to  $t$  and from  $e(\cdot, 0) = 0$ , we arrive at

$$\|\mathbf{e}_h\|^2 \leq 2c'h^{2s} \int_0^t \|y\|_{s+1}^2 dt + 2c_2 \int_0^t \|\mathbf{e}_h\|^2 dt \quad (36)$$

By Gronwall's inequality

$$\|\mathbf{e}_h\|^2 \leq ce^{ct} 2c'h^{2s} \int_0^t \|y\|_{s+1}^2 dt \quad (37)$$

The proof is completed.

Now, we shall derive an error estimate for the full discrete WG approximation.

**Theorem 4.4.** Let  $y \in H^{1+s}(\Omega)$  and  $y_h^n$  be the solutions of equations (13)–(14) and (28), respectively. Denote by  $\mathbf{e}^n := y_h^n - Q_h y(t_n)$  the difference between the full discrete WG approximation and the  $L^2$  projection of the exact solution  $y$ .

Then, there exists a constant  $C$  such that

$$\|\mathbf{e}^n\|^2 + \tau \sum_{i=1}^n \|\text{rot}_w \mathbf{e}^i\|^2 \leq \|\mathbf{e}^0\|^2 + C \left( h^{2r} \cdot \tau \|y\|_{1+r}^2 + \tau^2 \int_0^{t_n} \left\| \frac{\partial^2 y(s)}{\partial t^2} \right\|^2 ds \right) \quad (n > 0) \quad (38)$$

where  $y_h^n = y_h(t_n) = \{u_h(t_n), B_h(t_n)\}$ ,  $Q_h y(t_n) = \{Q_h u, Q_h B\}$ .

**Proof.** From the equations (13) and (27),

$$(\dot{y}, z_0) + A(y, z) + G(y, y, z) = (F(J_c), z_0) \quad (39)$$

$$\left( \frac{y_h^n - y_h^{n-1}}{\tau}, z_0 \right) + A(y_h^n, z) + G_w(y_h^n, y_h^n, z) = (F(J_c), z_0) \quad (40)$$

Subtracting the equation (39) from the equation (40).

$$\left( \frac{y_h^n - y_h^{n-1}}{\tau} - \dot{y}(t_n), z_0 \right) + A(y_h^n, z) - A(y(t_n), z) = G(y(t_n), y(t_n), z) - G_w(y_h^n, y_h^n, z) \quad (41)$$

Writing the equation (41) similarly with the equation (34),

$$\begin{aligned} & \left( \frac{y_h^n - y_h^{n-1}}{\tau} - \frac{Q_h y(t_n) - Q_h y(t_{n-1})}{\tau}, z_0 \right) + \left( \frac{Q_h y(t_n) - Q_h y(t_{n-1})}{\tau} - \dot{y}(t_n), z_0 \right) + A(y_h^n, z) - A(y(t_n), z) \\ &= (\Pi_h(\text{rot}y(t_n)) - \mathbf{R}_h(\text{rot}y(t_n)), \text{rot}_w z) + G(y(t_n), y(t_n), z) - G_w(y_h^n, y_h^n, z) \end{aligned}$$

Rewriting the above equation,

$$\begin{aligned} & \left( \frac{y_h^n - y_h^{n-1}}{\tau} - \frac{Q_h y(t_n) - Q_h y(t_{n-1})}{\tau}, z_0 \right) + \left( \frac{Q_h y(t_n) - Q_h y(t_{n-1})}{\tau} - \dot{y}(t_n), z_0 \right) + A(y_h^n, z) - A(Q_h y(t_n), z) \\ &= (\Pi_h(\text{rot}y(t_n)) - \mathbf{R}_h(\text{rot}y(t_n)), \text{rot}_w z) + G(y(t_n), y(t_n), z) - G_w(y_h^n, y_h^n, z) \end{aligned}$$

Taking account of  $\mathbf{e}^n = y_h^n - Q_h y(t_n)$ , we obtain

$$\left( \frac{e^n - e^{n-1}}{\tau}, z_0 \right) + A(e^n, z) = \left( \dot{y}_h(t_n) - \frac{y(t_n) - y(t_{n-1})}{\tau}, z_0 \right) + (\Pi_h(\text{roty}(t_n)) - R_h(\text{roty}(t_n)), \text{rot}_w z) \\ + G(y(t_n), y(t_n), z) - G_w(y_h^n, y_h^n, z)$$

Also, let  $w_1^n = \dot{y}(t_n) - \frac{y(t_n) - y(t_{n-1})}{\tau}$ ,  $w_2^n = \Pi_h(\text{roty}(t_n)) - R_h(\text{roty}(t_n))$  and choosing the  $z = e^n$ ,

$$\left( \frac{e^n - e^{n-1}}{\tau}, e^n \right) + A(e^n, e^n) = (w_1^n, e^n) + (w_2^n, \text{rot}_w e^n) + G(y(t_n), y(t_n), e^n) - G_w(y_h^n, y_h^n, e^n)$$

Therefore, we have

$$\|e^n\|^2 - (e^{n-1}, e^n) + \tau \|\text{rot}_w e^n\|^2 \leq \tau \|w_1^n\| \cdot \|e^n\| + \tau \|w_2^n\| \cdot \|\nabla_w e^n\| + \tau G(Q_h y(t_n), Q_h y(t_n), e^n) - G_w(y_h^n, y_h^n, e^n) \\ = \tau \|w_1^n\| \cdot \|e^n\| + \tau \|w_2^n\| \cdot \|\nabla_w e^n\| + \tau G(e^n, Q_h y(t_n), e^n) \\ \leq \tau \|w_1^n\| \cdot \|e^n\| + \tau \|w_2^n\| \cdot \|\nabla_w e^n\| + c\tau |e^n|^{\frac{3}{4}} \cdot \|e^n\|^{\frac{1}{4}} \cdot |Q_h y(t_n)| \cdot |e^n|^{\frac{3}{4}} \cdot \|e^n\|^{\frac{1}{4}}$$

Then,

$$\|e^n\|^2 + \tau \|\text{rot}_w e^n\|^2 \leq (e^{n-1}, e^n) + \tau \|w_1^n\| \cdot \|e^n\| + \tau \|w_2^n\| \cdot \|\nabla_w e^n\| + c\tau |e^n|^{\frac{3}{2}} \cdot \|e^n\|^{\frac{1}{2}} \cdot |y(t_n)|$$

By the boundedness of  $y$  and the Poincare' inequality, it follows

$$\|e^n\|^2 + \tau \|\text{rot}_w e^n\|^2 \leq \frac{1}{2} \|e^{n-1}\|^2 + \frac{1}{2} \|e^n\|^2 + \tau (\|w_1^n\| + \|w_2^n\| + c_1) \cdot \|\text{rot}_w e^n\|.$$

Consequently,

$$\frac{1}{2} \|e^n\|^2 + \tau \|\text{rot}_w e^n\|^2 \leq \frac{1}{2} \|e^{n-1}\|^2 + \frac{\tau}{2} (\|w_1^n\| + \|w_2^n\| + c_1)^2 + \frac{\tau}{2} \|\text{rot}_w e^n\|^2.$$

That is,

$$\frac{1}{2} \|e^n\|^2 + \frac{\tau}{2} \|\text{rot}_w e^n\|^2 \leq \frac{1}{2} \|e^{n-1}\|^2 + \frac{c_2 \tau}{2} (\|w_1^n\|^2 + \|w_2^n\|^2 + c_1^2)$$

By repeated application,

$$\|e^n\|^2 + \frac{\tau}{2} \|\text{rot}_w e^n\|^2 \leq \|e^0\|^2 + c_2 \tau \left( \sum_{i=1}^n \|w_1^i\|^2 + \sum_{i=1}^n \|w_2^i\|^2 + c_1^2 \right) \quad (42)$$

By Lemma 4.1 and  $R_h(\text{roty}) = \text{rot}_w(Q_h y)$ ,

$$\|w_2^i\| \leq c_3 h^s \|y(t_i)\|_{s+1}.$$

From  $\tau w_1^i = \tau \frac{\partial y(t_i)}{\partial t} - (y(t_i) - y(t_{i-1})) = \int_{t_{i-1}}^{t_i} (t - t_{i-1}) \frac{\partial^2 y(t)}{\partial t^2} dt$ ,

$$w_1^i = \frac{1}{\tau} \int_{t_{i-1}}^{t_i} (t - t_{i-1}) \frac{\partial^2 y(t)}{\partial t^2} dt \quad (43)$$

Thus,

$$\|w_1^i\|^2 = \int_{\Omega} \left( \frac{1}{\tau} \int_{t_{i-1}}^{t_i} (t-t_{i-1}) \frac{\partial^2 y(t)}{\partial t^2} ds \right)^2 d\Omega \leq \frac{1}{\tau^2} \int_{\Omega} \left[ \int_{t_{i-1}}^{t_i} (t-t_{i-1})^2 dt \cdot \int_{t_{i-1}}^{t_i} \left( \frac{\partial^2 y(t)}{\partial t^2} \right)^2 dt \right] d\Omega \leq c_4 \tau \cdot \int_{t_{i-1}}^{t_i} \left\| \frac{\partial^2 y(t)}{\partial t^2} \right\|^2 dt \quad (44)$$

Then by substituting (4.15) into (4.13), we have

$$\|e^n\|^2 + \tau \sum_{i=1}^n \|rot_w e^i\|^2 \leq \|e^0\|^2 + C \left( h^{2r} \cdot \tau \|y\|_{1+r}^2 + \tau^2 \int_0^{t_n} \left\| \frac{\partial^2 y(t)}{\partial t^2} \right\|^2 ds \right) \quad (n > 0)$$

The proof is completed.

## 5. Numerical Experiment

In this section, we present the results of numerical experiment. We carry out benchmark test in 2D. We consider a simple problem for the incompressible viscous MHD equations with known analytical solution in 2D. We choose  $\Omega = [0, 1] \times [0, 1]$  and time interval  $I = [0, 1]$  with computational domain. Also we assume that problem (2.1)-(2.5) has an analytic solution, which is given by  $u = (e^{-t} \sin 2\pi x \cos 2\pi y, -e^{-t} \cos 2\pi x \sin 2\pi y)^T$ ,

$$B = (e^{-t} \sin 2\pi y, e^{-t} \sin 2\pi x)^T,$$

By simple computation, we use the parameters  $\gamma = \frac{1}{8\pi^2}$ ,  $\gamma_\mu = \frac{1}{4\pi^2}$ ,  $a = 1$  in the tests.

We choose uniform triangular mesh and let  $h = 1/N$  ( $N = 8, 16, 32, 64, 128$ ) be mesh sizes for triangular meshes.

Let  $u$  and  $u_h$  be the exact velocity and the WG finite element approximation,  $B$  and  $B_h$  be the exact magnetic field and the WG finite element approximation.

In the test,  $\tau = h$  and  $\tau = h^2$  are used to check the order of convergence with respect to time step size  $\tau$  and mesh size  $h$ . The results are shown in Table 1 and Table 2.

**Table 1.** Result of WG finite element method with  $\gamma = 1/8\pi^2$ ,  $\gamma_\mu = 1/4\pi^2$ ,  $a = 1$  and  $\tau = h$ .

$1/h$	$\ \nabla(u - u_h^n)\ $	$\ u - u_h^n\ $	$\ \nabla(B - B_h^n)\ $	$\ B - B_h^n\ $
8	1.054e-2	1.32e-3	3.78e-1	4.16e-3
16	3.42e-3	2.64e-4	6.15e-1	3.58e-4
32	2.609e-3	5.41e-4	2.79e-2	5.17e-4
64	4.27e-4	3.92e-5	4.38e-3	3.26e-5
128	8.11e-4	4.59e-6	7.43e-4	2.91e-5

**Table 2.** Result of WG finite element method with  $\gamma = 1/8\pi^2$ ,  $\gamma_\mu = 1/4\pi^2$ ,  $a = 1$  and  $\tau = h^2$ .

$1/h$	$\ \nabla(u - u_h^n)\ $	$\ u - u_h^n\ $	$\ \nabla(B - B_h^n)\ $	$\ B - B_h^n\ $
8	2.37e-3	1.74e-5	4.05e-2	2.95e-4
16	6.27e-4	3.89e-6	2.71e-2	5.38e-5
32	3.19e-4	7.15e-6	3.83e-3	4.79e-5
64	4.08e-5	2.57e-6	1.82e-4	7.48e-6
128	3.93e-5	5.29e-7	2.36e-5	6.83e-6

From these results, we can know that WG finite element method is advisable and efficient.

Remark 2. For the above exact solutions  $u$  and  $B$ , we know that hydrodynamic pressure  $p$  and minor control possible electromotive force  $J_c$  are  $p = e^{-2t} (\cos^2 2\pi x + \cos^2 2\pi y - \cos 2\pi x \cdot \cos 2\pi y)$ ,  $J_c = -e^{-2t} (\sin^2 2\pi x \cdot \cos 2\pi y + \cos 2\pi x \cdot \sin^2 2\pi y)$ , respectively.

## 6. Conclusion

In this paper, we have formulated the weak Galerkin finite element scheme for the incompressible viscous magneto-hydrodynamic equations on arbitrary polygons or polyhedra with certain shape regularity. Also, we have estimated the error of the semi-discrete and full-discrete approximate solutions by the weak Galerkin finite element method for the incompressible viscous Magneto-hydrodynamic equations.

In particular, we have derived the optimal error estimates for velocity and magnetic field in Hilbert space, respectively.

In future work, we will develop for more general problems, such as incompressible viscous Magneto-hydrodynamic equations with other boundary conditions and numerical experiment et al, from practical requirement.

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